CHAPTER 2

2.1) Consider the momentum vector

$$p^\mu = (E,\ E\sin\theta\cos\phi,\ E\sin\theta\sin\phi,\ E\cos\theta)$$

Express $p_{a\dot b}$, $p^{\dot a b}$ in terms of E, $\sin\theta/2$, $\cos\theta/2$, & $e^{\pm i\phi}$.

$$p_{a\dot b} = p_\mu(\sigma^\mu)_{a\dot b} = E\left(-\mathbb{1} + \sin\theta\cos\phi\begin{pmatrix}0&1\\1&0\end{pmatrix} + \sin\theta\sin\phi\begin{pmatrix}0&-i\\i&0\end{pmatrix} + \cos\theta\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)$$

$$= E\begin{pmatrix}-1+\cos\theta & \sin\theta(\cos\phi - i\sin\phi)\\ \sin\theta(\cos\phi+i\sin\phi) & -1-\cos\theta\end{pmatrix} = 2E\begin{pmatrix}-\sin^2\theta/2 & \sin\theta/2\cos\theta/2\, e^{-i\phi}\\ \sin\theta/2\cos\theta/2\, e^{i\phi} & -\cos^2\theta/2\end{pmatrix}$$

$$p^{\dot a b} = p_\mu(\bar\sigma^\mu)^{\dot a b} = E\left(-\mathbb{1} - \sin\theta\cos\phi\begin{pmatrix}0&1\\1&0\end{pmatrix} - \sin\theta\sin\phi\begin{pmatrix}0&-i\\i&0\end{pmatrix} - \cos\theta\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)$$

$$= E\begin{pmatrix}-1-\cos\theta & -\sin\theta(\cos\phi - i\sin\phi)\\ -\sin\theta(\cos\phi+i\sin\phi) & -1+\cos\theta\end{pmatrix} = -2E\begin{pmatrix}\cos^2\theta/2 & \sin\theta/2\cos\theta/2\, e^{-i\phi}\\ \sin\theta/2\cos\theta/2\, e^{i\phi} & \sin^2\theta/2\end{pmatrix}$$

Show that the helicity spinor $|p\rangle^{\dot a} = \sqrt{2E}\begin{pmatrix}\cos\theta/2\\ \sin\theta/2\, e^{i\phi}\end{pmatrix}$ solves the massless Weyl eqn.

$$p_{a\dot b}|p\rangle^{\dot b} = E\sqrt{2E}\begin{pmatrix}-2\sin^2\theta/2 & \sin\theta\, e^{-i\phi}\\ \sin\theta\, e^{i\phi} & -2\cos^2\theta/2\end{pmatrix}\begin{pmatrix}\cos\theta/2\\ \sin\theta/2\, e^{i\phi}\end{pmatrix}$$

$$\sim \begin{pmatrix}-2\sin^2\theta/2\cos\theta/2 + \sin\theta\sin\theta/2\\ (\sin\theta\cos\theta/2 - 2\cos^2\theta/2\sin\theta/2)e^{i\phi}\end{pmatrix} = \begin{pmatrix}-\sin\theta\sin\theta/2 + \sin\theta\sin\theta/2\\ (\sin\theta\cos\theta/2 - \sin\theta\cos\theta/2)e^{i\phi}\end{pmatrix} = \vec{0}$$

Find expressions for the spinors $\langle p|_{\dot a}$, $|p]_a$, & $[p|^a$ and check that they satisfy $p_{a\dot b} = -|p]_a\langle p|_{\dot b}$, $p^{\dot a b} = -|p\rangle^{\dot a}[p|^b$

Since $p^\mu \in \mathbb{R}^4$, we can use $[p|^a = (|p\rangle^{\dot a})^*$.

$$\Rightarrow [p|^a = \sqrt{2E}\begin{pmatrix}\cos\theta/2\\ \sin\theta/2\, e^{-i\phi}\end{pmatrix}$$

Raising & lowering w/ ϵ^{ab}:

$$|p]_a = \epsilon_{ab}[p|^b = \begin{pmatrix}0&-1\\1&0\end{pmatrix}\sqrt{2E}\begin{pmatrix}\cos\theta/2\\ \sin\theta/2\, e^{-i\phi}\end{pmatrix} = \sqrt{2E}\begin{pmatrix}-\sin\theta/2\, e^{-i\phi}\\ \cos\theta/2\end{pmatrix}$$

$$\langle p|_{\dot a} = \epsilon_{\dot a\dot b}|p\rangle^{\dot b} = \begin{pmatrix}0&-1\\1&0\end{pmatrix}\sqrt{2E}\begin{pmatrix}\cos\theta/2\\ \sin\theta/2\, e^{i\phi}\end{pmatrix} = \sqrt{2E}\begin{pmatrix}-\sin\theta/2\, e^{i\phi}\\ \cos\theta/2\end{pmatrix}$$

$$-|p]_a\langle p|_{\dot b} = -2E\begin{pmatrix}-\sin\theta/2\, e^{-i\phi}\\ \cos\theta/2\end{pmatrix}\begin{pmatrix}-\sin\theta/2\, e^{i\phi} & \cos\theta/2\end{pmatrix}$$

$$= 2E\begin{pmatrix}-\sin^2\theta/2 & \sin\theta/2\cos\theta/2\, e^{-i\phi}\\ \cos\theta/2\sin\theta/2\, e^{i\phi} & -\cos^2\theta/2\end{pmatrix} = p_{a\dot b}$$

$$-|p\rangle^{\dot a}[p|^b = -2E\begin{pmatrix}\cos\theta/2\\ \sin\theta/2\, e^{i\phi}\end{pmatrix}\begin{pmatrix}\cos\theta/2 & \sin\theta/2\, e^{-i\phi}\end{pmatrix}$$

$$= -2E\begin{pmatrix}\cos^2\theta/2 & \sin\theta/2\cos\theta/2\, e^{-i\phi}\\ \sin\theta/2\cos\theta/2\, e^{i\phi} & \sin^2\theta/2\end{pmatrix} = p^{\dot a b}$$

2.2) The helicity of a massless particle is the projection of the spin along the momentum 3-vector $\vec{p}$, so the helicity op. can be written $\Sigma = S\cdot\frac{\vec{p}}{|\vec{p}|}$, where the spin $S_i = \frac{1}{2}\epsilon_{ijk}S^{jk}$ $(i,j,k=1,2,3)$ is defined by the spin matrix $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu,\gamma^\nu]$. $\quad \epsilon_{123}S^{23}+\epsilon_{132}S^{32}$

Pick a frame where p^μ is along $\hat{z}$. Use Ex (2.1) to show that the chiral basis $\quad \gamma^\mu = \begin{pmatrix}0 & (\sigma^\mu)_{a\dot b}\\ (\bar\sigma^\mu)^{\dot a b} & 0\end{pmatrix}$

$$\begin{cases} v_+(p) = \begin{pmatrix}|p]_a\\ 0\end{pmatrix}, \quad v_-(p) = \begin{pmatrix}0\\ |p\rangle^{\dot a}\end{pmatrix}\\ \bar u_-(p) = (0 \quad \langle p|_{\dot a}), \quad \bar u_+(p) = ([p|^a \quad 0)\end{cases}$$

is also a helicity basis, i.e. $\Sigma v_\pm = -h_\pm v_\pm$ for $h_\pm = \pm\frac{1}{2}$.

$$S_i = \frac{1}{2}\epsilon_{ijk}S^{jk} = \frac{i}{4}\epsilon_{ijk}[\gamma^j,\gamma^k] = \frac{-i}{4}\epsilon_{ijk}\begin{pmatrix}[\sigma^j,\sigma^k] & 0\\ 0 & [\sigma^j,\sigma^k]\end{pmatrix}$$

$$= \frac{-i}{4}\epsilon_{ijk}\begin{pmatrix}2i\epsilon_{jkl}\sigma^l & 0\\ 0 & 2i\epsilon_{jkl}\sigma^l\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}$$

let $p^\mu = (E,0,0,E)$ $\Rightarrow \vec{p} = (0,0,E)$ $\quad |\vec{p}| = E \quad (\theta=0,\ \phi=0)$

$$\Sigma = S_i\frac{p^i}{E} = \frac{1}{2}\begin{pmatrix}0 & \sigma^3\\ \sigma^3 & 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1&0&&\\0&-1&&\\&&1&0\\&&0&-1\end{pmatrix}$$

$$|p]_a = \sqrt{2E}\begin{pmatrix}0\\1\end{pmatrix}, \quad |p\rangle^{\dot a} = \sqrt{2E}\begin{pmatrix}1\\0\end{pmatrix}$$

$$v_+(p) = \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad v_-(p) = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$$\Sigma v_+(p) = \frac{\sqrt{2E}}{2}\begin{pmatrix}1&0&&\\0&-1&&\\&&1&0\\&&0&-1\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \frac{\sqrt{2E}}{2}\begin{pmatrix}0\\-1\\0\\0\end{pmatrix} = -\frac{1}{2}v_+(p)$$

$$\Sigma v_-(p) = \frac{\sqrt{2E}}{2}\begin{pmatrix}1&0&&\\0&-1&&\\&&1&0\\&&0&-1\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \frac{\sqrt{2E}}{2}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \frac{1}{2}v_-(p)$$

2.#) Derive the relation $\langle pq\rangle[pq] = 2p\cdot q = (p+q)^2$.

$$\langle pq\rangle[pq] = \langle p|_{\dot a}|q\rangle^{\dot a}[p|^a|q]_a = -\langle p|_{\dot a}|q\rangle^{\dot a}[q|^a|p]_a = -p_{a\dot a}q^{\dot a a}$$

$$= -p_\mu q_\nu(\sigma^\mu)_{a\dot a}(\bar\sigma^\nu)^{\dot a a} = -p_\mu q_\nu Tr(\sigma^\mu\bar\sigma^\nu) = 2p_\mu q_\nu\eta^{\mu\nu} = 2p\cdot q$$

$(p+q)^2 = \cancelto{0}{p^2} + 2p\cdot q + \cancelto{0}{q^2}$ since p, q are for massless particles

2.3) Prove the Fierz identity $\langle 1|\gamma^\mu|2]\langle 3|\gamma_\mu|4] = 2\langle 13\rangle[24]$.

$$(0\ \ \langle 1|_{\dot a})\begin{pmatrix}0 & (\sigma^\mu)_{a\dot b}\\ (\bar\sigma^\mu)^{\dot a b} & 0\end{pmatrix}\begin{pmatrix}|2]_b\\0\end{pmatrix} = \langle 1|_{\dot a}(\bar\sigma^\mu)^{\dot a b}|2]_b$$

$$\langle 1|\gamma^\mu|2]\langle 3|\gamma_\mu|4] = \langle 1|_{\dot a}(\bar\sigma^\mu)^{\dot a b}|2]_b\langle 3|_{\dot b}(\bar\sigma_\mu)^{\dot b a}|4]_a$$

$$= -2\langle 1|_{\dot a}\epsilon^{\dot a\dot b}|2]_b\langle 3|_{\dot b}\epsilon^{ba}|4]_a = -2\langle 1|_{\dot a}|3\rangle^{\dot a}[4|^b|2]_b$$

$$= -2\langle 13\rangle[42] = 2\langle 13\rangle[24]$$

2.4) Show that $\langle k|\gamma^\mu|k] = 2k^\mu$ & $\langle k|P|k] = 2P\cdot k$.

$$\langle k|\gamma^\mu|k] = \langle k|_{\dot a}(\bar\sigma^\mu)^{\dot a b}|k]_b = -p_{b\dot a}(\bar\sigma^\mu)^{\dot a b} = -k_\nu(\sigma^\nu)_{b\dot a}(\bar\sigma^\mu)^{\dot a b} = -k_\nu\cdot -2\eta^{\mu\nu} = 2k^\mu$$

$$\langle k|P|k] = P_\mu\langle k|\gamma^\mu|k] = 2P_\mu k^\mu = 2P\cdot k$$

2.5) Use standard techniques to show that $\langle |A_4(\phi\bar f f\phi)|^2\rangle = 2g^4\frac{(s-t)^2}{st}$

$$\langle |A_4(\phi\bar f^{h_2}f^{h_3}\phi)|^2\rangle = \sum_{h_2,h_3}|A_4(\phi\bar f^{h_2}f^{h_3}\phi)|^2$$

$iA_4(\phi\bar f^{h_2}f^{h_3}\phi) =$ 

$+ (1\leftrightarrow 4) = (ig)^2\bar u_{h_3}(p_3)\frac{-i(\not p_1+\not p_2)}{(p_1+p_2)^2}v_{h_2}(p_2) + (1\leftrightarrow 4)$

$(p_2+p_4)^2 = (p_1+p_3)^2 = -t$

if $h_2 = h_3$, it's 0.

$$iA_4(h_2=\pm, h_3=\mp) = (-ig^2)\bar u_\pm(p_3)\frac{i(\not p_1+\not p_2)}{s}v_\mp(p_2) = (-ig^2)\bar u_\pm(p_3)\frac{i\not p_1}{s}v_\mp(p_2) + (1\leftrightarrow 4)$$

$$= (-ig^2)\bar u_\pm(p_3)\frac{i\not p_1}{s}u_\pm(p_2) + (1\leftrightarrow 4) = g^2\bar u_\pm(p_3)\left(\frac{\not p_1}{s}+\frac{\not p_4}{t}\right)u_\pm(p_2)$$

$$\sum_{h_2=\pm}|A_4|^2 = \sum_{\substack{s,s'\\ r,r'}} g^4\left(\bar u_s(p_3)A_{sr}u_r(p_2)\right)\left(\bar u_{s'}(p_2)A_{s'r'}u_{r'}(p_3)\right)\delta_{sr}\delta_{s'r'} \qquad \left(A = \frac{\not p_1}{s}+\frac{\not p_4}{t} = \left(\frac{p_1^\mu}{s}+\frac{p_4^\mu}{t}\right)\gamma_\mu = \frac{p_1^\mu t+p_4^\mu s}{st}\gamma_\mu\right)$$

$$= \sum_{\substack{s,s'\\ r,r'}} g^4\, u_{r'}(p_3)\bar u_s(p_3)A_{sr}u_r(p_2)\bar u_{s'}(p_2)A_{s'r'}$$

$$= g^4\,\mathrm{Tr}\left(u(p_3)\bar u(p_3)A\,u(p_2)\bar u(p_2)A\right) = g^4\,\mathrm{Tr}\left(\not p_3 A \not p_2 A\right)$$

$$= \frac{g^4}{(st)^2}\mathrm{Tr}\left(p_{3\mu}\gamma^\mu(p_{1\nu}t+p_{4\nu}s)\gamma^\nu p_{2\rho}\gamma^\rho(p_{1\sigma}t+p_{4\sigma}s)\gamma^\sigma\right)$$

$$= \frac{g^4}{(st)^2}p_{3\mu}(p_{1\nu}t+p_{4\nu}s)p_{2\rho}(p_{1\sigma}t+p_{4\sigma}s)\underbrace{\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right)}_{4(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\rho})}$$

$$= \frac{4g^4}{(st)^2}\Big(2(p_3\cdot p_1 t+p_3\cdot p_4 s)(p_2\cdot p_1 t+p_2\cdot p_4 s) - (p_3\cdot p_2)(p_1^2t^2+2p_1\cdot p_4 st+p_4^2s^2)\Big)$$

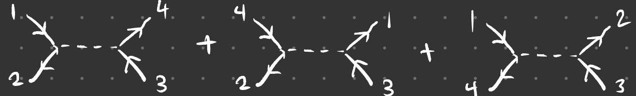
$$= \frac{4g^4}{(st)^2}\left[2\left(\frac{t-p_1^2-p_3^2}{2}t+\frac{s-p_3^2-p_4^2}{2}s\right)\left(\frac{s-p_1^2-p_2^2}{2}t+\frac{t-p_2^2-p_4^2}{2}s\right) - \left(\frac{u-p_2^2-p_3^2}{2}\right)\left(p_1^2t^2+(u-p_1^2-p_4^2)st+p_4^2s^2\right)\right]$$

$$= \frac{2g^4}{(st)^2}\left((t^2+s^2)2st-u^2st\right) = \frac{2g^4}{st}\left(2t^2+2s^2-(s+t)^2\right)$$

$$= \frac{2g^4}{st}(t^2+s^2-2st) = \frac{2g^4}{st}(s-t)^2$$

2.6) Calculate the 4-fermion "all minus" amplitude $A_4(\bar f^- f^- \bar f^- f^-)$ in Yukawa theory.

$$iA_4(\bar f^- f^- \bar f^- f^-) =$$



$$= ig^2\left(\bar u_4 v_3\frac{1}{(p_1+p_2)^2}\bar u_2 v_1 + (1\leftrightarrow 4) + (1\leftrightarrow 3)\right)$$

$$= ig^2\left(\frac{\langle 34\rangle\langle 21\rangle}{\langle 34\rangle[34]}+\frac{\langle 31\rangle\langle 24\rangle}{\langle 24\rangle[24]}+\frac{\langle 14\rangle\langle 23\rangle}{\langle 23\rangle[23]}\right)$$

$$= -ig^2\left(\frac{\langle 12\rangle}{[34]}+\frac{\langle 13\rangle}{[24]}+\frac{\langle 14\rangle}{[23]}\right)$$

2.7) Calculate the spin sum $\langle |\mathcal{A}_4(\bar{f}f\bar{f}f)|^2\rangle$ for the 4-fermion process in Yukawa theory.

$i\mathcal{A}_4(\bar{f}^{h_1}f^{h_2}\bar{f}^{h_3}f^{h_4}) = 0$ unless $h_1 = h_2$ & $h_3 = h_4$

$$\langle |\mathcal{A}_4(\bar{f}f\bar{f}f)|^2\rangle = |\mathcal{A}_4(\bar{f}^+f^+\bar{f}^+f^+)|^2 + |\mathcal{A}_4(++--)|^2 + |\mathcal{A}_4(--++)|^2 + |\mathcal{A}_4(----)|^2$$

$$i\mathcal{A}_4(++++) = -ig^2\left(\frac{[12]}{\langle 34\rangle} + \frac{[13]}{\langle 24\rangle} + \frac{[14]}{\langle 23\rangle}\right) = -ig^2\left(\frac{[34]}{\langle 12\rangle} + \frac{[24]}{\langle 13\rangle} + \frac{[23]}{\langle 14\rangle}\right)$$

$$i\mathcal{A}_4(++--) = ig^2\frac{\langle 34\rangle}{\langle 12\rangle} = \frac{1}{i\mathcal{A}_4(--++)}$$

$$\langle |\mathcal{A}_4(\bar{f}f\bar{f}f)|^2\rangle = g^4\left[\left(\frac{\langle 12\rangle}{[34]} + \frac{\langle 13\rangle}{[24]} + \frac{\langle 14\rangle}{[23]}\right)^2 + \left(\frac{[34]}{\langle 12\rangle} + \frac{[24]}{\langle 13\rangle} + \frac{[23]}{\langle 14\rangle}\right)^2 + \left(\frac{\langle 12\rangle}{\langle 34\rangle}\right)^2 + \left(\frac{\langle 34\rangle}{\langle 12\rangle}\right)^2\right]$$

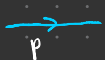
2.8) Consider a model w/ a Weyl fermion ψ & a complex scalar ϕ:

$$\mathcal{L} = i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi - \partial_\mu\bar{\phi}\partial^\mu\phi + \tfrac{1}{2}g\phi\psi\psi + \tfrac{1}{2}g^*\bar{\phi}\psi^\dagger\psi^\dagger - \tfrac{1}{4}\lambda|\phi|^4$$

Show that $\mathcal{A}_4(\phi\phi\bar{\phi}\bar{\phi}) = -\lambda$, $\mathcal{A}_4(\phi f^- f^+\bar{\phi}) = -|g|^2\frac{\langle 24\rangle}{\langle 34\rangle}$, &

$\mathcal{A}_4(f^-f^-f^+f^+) = |g|^2\frac{\langle 12\rangle}{\langle 34\rangle}$.

Feynman rules:

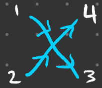
 $= \frac{-i}{p^2}$ ϕ propagator

 $= \frac{i\not{p}}{p^2}$ ψ propagator

 $= -i\lambda$

 $= ig^*$

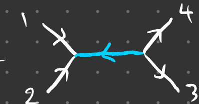
 $= ig$

$iA_4(\phi\phi\bar\phi\bar\phi) =$  $= -i\lambda \quad \Rightarrow A_4(\phi\phi\bar\phi\bar\phi) = -\lambda$

$iA_4(\phi f^- f^+ \bar\phi) =$  $= -|g|^2\, \bar u_{2-} \frac{i(\not p_1+\not p_2)}{(p_1+p_2)^2} v_{3+}$

$$= i|g|^2 \frac{\langle 2|(3+4)|3]}{\langle 34\rangle[34]} = -i|g|^2\frac{\langle 24\rangle[34]}{\langle 34\rangle[34]} = -i|g|^2\frac{\langle 24\rangle}{\langle 34\rangle}$$

$$\Rightarrow A_4(\phi f^- f^+\bar\phi) = -|g|^2 \frac{\langle 24\rangle}{\langle 34\rangle}$$

$iA_4(f^-f^-f^+f^+) =$  $= -|g|^2[43]\frac{-i}{\langle 34\rangle[34]}\langle 21\rangle = i|g|^2\frac{\langle 12\rangle}{\langle 34\rangle}$

$$\Rightarrow A_4(f^-f^-f^+f^+) = |g|^2\frac{\langle12\rangle}{\langle34\rangle}$$

2.9) Show that $A_5(f^-\bar f^-\phi\phi\phi) = g^3\frac{[12][34]^2}{[13][14][23][24]} + (3\leftrightarrow5) + (4\leftrightarrow5)$ in Yukawa theory.

$iA_5(f^-\bar f^-\phi\phi\phi) =$  $+ (3\leftrightarrow5) + (4\leftrightarrow5)$

$$= (ig)^3\, \bar u_{1-}\frac{-i(\not p_1+\not p_3)}{(p_1+p_3)^2}\,\frac{-i(\not p_2+\not p_5)}{(p_2+p_5)^2}\, v_{2-} = ig^3\langle 1|\frac{(\cancel{|1]\langle 1|}+|3\rangle[3|)(\cancel{|2]\langle 2|}+|5]\langle 5|)}{\langle13\rangle[13]\langle25\rangle[25]}|2\rangle$$

$$= ig^3\frac{\cancel{\langle13\rangle}[35]\langle52\rangle}{\cancel{\langle13\rangle}[13]\langle25\rangle[25]} = -ig^3\frac{[35]}{[13][25]}$$

$4\leftrightarrow5$: $\frac{[34]}{[13][24]} = \frac{[12][34]^2}{[12][34][13][24]} = \frac{[12][34]^2}{([13][24]+[14][23])[13][24]}$

$[12][34] = [13][24]+[14][23]$ $\quad \langle 1|35|2\rangle$

FIX THIS

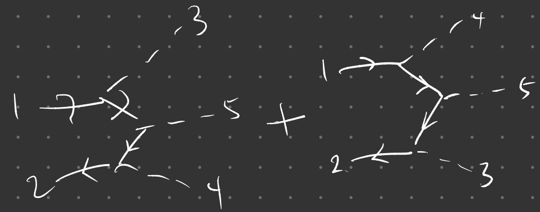
$$\gamma^\mu\gamma^\nu = \begin{pmatrix}0 & (\sigma^\mu)_{a\dot b}\\ (\bar\sigma^\mu)^{\dot a b} & 0\end{pmatrix}\begin{pmatrix}0&\sigma^\nu\\ \bar\sigma^\nu & 0\end{pmatrix} = \begin{pmatrix}\sigma^\mu\bar\sigma^\nu & 0\\ 0 & \bar\sigma^\mu\sigma^\nu\end{pmatrix}$$

$\langle 1|(1+3)(2+5)|2\rangle$

$(1+3)(2+5)$

$= (\cancel{12}+\cancel{15}+\cancel{32}+35)$

$$\langle 1|^a(\not 3)_{ab}(\not 5)^{bc}|2\rangle_c$$
$$= p_{3\mu}p_{5\nu}\langle 1|^a(\gamma^\mu)_{ab}(\gamma^\nu)^{bc}|2\rangle_c$$



$$= \frac{[34]}{[13][24]} + \frac{[34]}{[14][23]} = \frac{\left([14][23] + [13][24]\right)[34]}{[13][14][23][24]} = \frac{[12][34]^2}{[13][14][23][24]}$$

2.10) Consider the momentum $p^\mu = (E, E\sin\theta\cos\phi, E\sin\theta\sin\phi, E\cos\theta)$ & the polarization vectors

$$\tilde\epsilon^\mu_\pm(p) = \pm\frac{e^{\mp i\phi}}{\sqrt{2}}\left(0, \cos\theta\cos\phi \pm i\sin\phi, \cos\theta\sin\phi \mp i\cos\phi, -\sin\theta\right).$$

Note that for $\theta=\phi=0$, $\tilde\epsilon^\mu_\pm = \pm\frac{1}{\sqrt{2}}(0, 1, \mp i, 0)$.

a) Show that $\tilde\epsilon_\pm(p)^2 = 0$ & $\tilde\epsilon_\pm(p)\cdot p = 0$.

$$\tilde\epsilon_\pm(p)^2 \sim (\cos\theta\cos\phi \pm i\sin\phi)^2 + (\cos\theta\sin\phi \mp i\cos\phi)^2 + \sin^2\theta$$

$$= \cos^2\theta\cos^2\phi \pm 2i\cos\theta\cancel{\cos\phi}\sin\phi - \sin^2\phi + \cos^2\theta\sin^2\phi \mp 2i\cos\cancel{\theta\sin\phi}\cos\phi$$
$$-\cos^2\phi + \sin^2\theta$$

$$= 0$$

$$\tilde\epsilon_\pm(p)\cdot p \sim \sin\theta\cos\phi(\cos\theta\cancel{\cos\phi} \pm i\sin\phi) + \sin\theta\sin\phi(\cos\cancel{\theta}\sin\phi \mp i\cos\phi)$$
$$-\sin\cancel{\theta}\cos\theta$$

$$= 0$$

b) Since $\tilde\epsilon^\mu_\pm(p)$ is null, $(\tilde\epsilon^\mu_\pm(p))_{a\dot b} = (\sigma_\mu)_{a\dot b}\tilde\epsilon^\mu_\pm(p)$ can be written as a product of a square & an angle spinor. To see this specifically, first calculate $(\tilde\epsilon^\mu_\pm(p))_{a\dot b}$, then find a $\langle r|$ s.t. $(\tilde\epsilon^\mu_+(p))_{a\dot b} = -|p]_a\langle r|_{\dot b}$

$$(\tilde\epsilon_\pm(p))_{a\dot b} = \pm\frac{e^{\mp i\phi}}{\sqrt{2}}\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}(\cos\theta\cos\phi \pm i\sin\phi) + \begin{pmatrix}0&-i\\i&0\end{pmatrix}(\cos\theta\sin\phi \mp i\cos\phi)\right.$$
$$\left. - \begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\theta\right)$$

$$= \pm\frac{e^{\mp i\phi}}{\sqrt{2}}\begin{pmatrix}-\sin\theta & \cos\theta\cos\phi \pm i\sin\phi - i\cos\theta\sin\phi \mp \cos\phi \\ \cos\theta\cos\phi \pm i\sin\phi + i\cos\theta\sin\phi \pm \cos\phi & \sin\theta\end{pmatrix}$$

$$= \pm\frac{e^{\mp i\phi}}{\sqrt{2}}\begin{pmatrix}-\sin\theta & \cos\theta e^{-i\phi} \mp e^{-i\phi} \\ \cos\theta e^{i\phi} \pm e^{i\phi} & \sin\theta\end{pmatrix}$$

$$(\tilde\epsilon_+(p))_{a\dot b} = \frac{e^{-i\phi}}{\sqrt{2}}\begin{pmatrix}-\sin\theta & -2\sin^2\theta/2\, e^{-i\phi} \\ 2\cos^2\theta/2\, e^{i\phi} & \sin\theta\end{pmatrix}$$

$$= \sqrt{2}\begin{pmatrix}-\sin\theta/2\cos\theta/2\, e^{-i\phi} & -\sin^2\theta/2\, e^{-2i\phi} \\ \cos^2\theta/2 & \sin\theta/2\cos\theta/2\, e^{-i\phi}\end{pmatrix}$$

$$= -\sqrt{2E}\begin{pmatrix} -\sin\theta/2\, e^{-i\phi} \\ \cos\theta/2 \end{pmatrix}(a \quad b) = \sqrt{2E}\begin{pmatrix} a\sin\theta/2\, e^{-i\phi} & b\sin\theta/2\, e^{-i\phi} \\ -a\cos\theta/2 & -b\cos\theta/2 \end{pmatrix}$$

$$a = \frac{-\cos\theta/2}{\sqrt{E}}, \quad b = \frac{-\sin\theta/2}{\sqrt{E}} e^{-i\phi}$$

$$\langle r|_{\dot{b}} = \frac{-1}{\sqrt{E}}\left(\cos\theta/2 \quad \sin\theta/2\, e^{-i\phi}\right)$$

$$\langle rp\rangle = -\frac{1}{\sqrt{E}}\sqrt{2E}\left(\cos\theta/2 \quad \sin\theta/2\, e^{-i\phi}\right)\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2\, e^{i\phi} \end{pmatrix} = -\sqrt{2}$$

c) Next, show that $(\epsilon_+(p;q))_{a\dot{b}} = \frac{\sqrt{2}}{\langle qp\rangle}|p]\langle q|$.

$$(\epsilon_+(p;q))_{a\dot{b}} = (\sigma_\mu)_{a\dot{b}}\epsilon_+^\mu(p;q) = -(\sigma_\mu)_{a\dot{b}}\frac{\langle q|\gamma^\mu|p]}{\sqrt{2}\langle qp\rangle} = -(\sigma_\mu)_{a\dot{b}}\frac{\langle q|_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b}|p]_b}{\sqrt{2}\langle qp\rangle}$$

$$= -\frac{1}{\sqrt{2}\langle qp\rangle}\langle q|_{\dot{a}}(\sigma_\mu)_{\dot{a}b}\epsilon^{bc}\epsilon^{\dot{a}\dot{c}}(\sigma^\mu)_{c\dot{c}}|p]_b = \frac{\sqrt{2}}{\langle qp\rangle}\langle q|_{\dot{a}}\epsilon_{ac}\epsilon_{\dot{b}\dot{c}}\epsilon^{bc}\epsilon^{\dot{a}\dot{c}}|p]_b$$

$$= \frac{\sqrt{2}}{\langle qp\rangle}\delta_a{}^b\delta_{\dot{b}}{}^{\dot{a}}\langle q|_{\dot{a}}|p]_b = \frac{\sqrt{2}}{\langle qp\rangle}\langle q|_{\dot{b}}|p]_a = \frac{\sqrt{2}}{\langle qp\rangle}|p]\langle q|$$

d) Now suppose there is a constant c_+ s.t. $\epsilon_+^\mu(p;q) = \tilde{\epsilon}_+^\mu(p) + c_+p^\mu$. Show that this condition requires $\langle rp\rangle = -\sqrt{2}$, then show that $c_+ = \frac{-\langle rq\rangle}{\langle pq\rangle}$.

$$(\epsilon_+(p;q))_{a\dot{b}} = (\tilde{\epsilon}_+(p))_{a\dot{b}} + c_+\overbrace{(\sigma_\mu)_{a\dot{b}}p^\mu}^{p_{a\dot{b}}} = -|p]_a\langle r|_{\dot{b}} - c_+|p]_a\langle p|_{\dot{b}}$$

$$= \frac{\sqrt{2}}{\langle qp\rangle}|p]_a\langle q|_{\dot{b}}$$

$$[qp]\langle rp\rangle = \frac{-\sqrt{2}}{\langle qp\rangle}[qp]\langle qp\rangle \Rightarrow \langle rp\rangle = -\sqrt{2}$$

$$c_+[qp]\langle pr\rangle = \frac{-\sqrt{2}}{\langle qp\rangle}[qp]\langle qr\rangle \Rightarrow c_+ = \frac{-\sqrt{2}}{\langle qp\rangle}\frac{\langle qr\rangle}{\langle pr\rangle} = \frac{-\sqrt{2}}{\sqrt{2}}\frac{\langle rq\rangle}{\langle pq\rangle} = -\frac{\langle rq\rangle}{\langle pq\rangle}$$

2.11) Show that $\mathcal{A}_4(\bar{f}^+f^-\gamma^-\gamma^-) = 0$ in QED w/o making any choice of the reference spinors q_3, q_4.

$$i\mathcal{A}_4(\bar{f}^+f^-\gamma^-\gamma^-) = (ie^2)\bar{u}_{2-}\not{\epsilon}_{4-}\frac{-i(\not{p}_1+\not{p}_3)}{(p_1+p_3)^2}\not{\epsilon}_{3-}v_{1+} + (3 \leftrightarrow 4)$$

$$\sim \langle 2| \frac{1}{[q_4 4]} \left(|4\rangle [q_4| + \cancel{|q_4]\langle 4|}^{\,0}\right) \frac{(1+3)}{\langle 13\rangle [13]} \frac{1}{[q_3 3]} \left(|3\rangle [q_3| + \cancel{|q_3]\langle 3|}^{\,0}\right) |1] + (3 \leftrightarrow 4)$$

$$= \frac{1}{[q_4 4][q_3 3]\langle 13\rangle [13]} \langle 24\rangle [q_4|(1+3)|3\rangle [q_3 1] + (3 \leftrightarrow 4)$$

$$= \frac{\langle 24\rangle [q_4|1|3\rangle [q_3 1]}{[q_4 4][q_3 3]\langle 13\rangle [13]} + (3 \leftrightarrow 4) = -\frac{\langle 24\rangle \langle 31\rangle [1 q_4][q_3 1]}{[q_4 4][q_3 3]\langle 13\rangle [13]} + (3 \leftrightarrow 4)$$

$$= -\frac{\langle 24\rangle \langle q_4 1\rangle [q_3 1]}{[q_4 4][q_3 3][13]} - \frac{\langle 23\rangle \langle q_3 1\rangle [q_4 1]}{[q_3 3][q_4 4][14]}$$

$$= -\left(\frac{\langle 24\rangle \langle q_4 1\rangle [q_3 1][14] + \langle 23\rangle \langle q_3 1\rangle [q_4 1][13]}{[q_4 4][q_3 3][13][14]}\right)$$

$$\sim \langle q_4 1\rangle [q_3 1] - \langle q_3 1\rangle [q_4 1] = -\langle q_4 1\rangle [1 q_3] + \langle q_3 1\rangle [1 q_4]$$

$$= \langle q_4|1|q_3] - \langle q_3|1|q_4]$$

STUCK :(

2.12) Show that $A_4(\bar{f}^+ f^- \gamma^+ \gamma^-)$ is independent of q_3, q_4.

$$A_4(\bar{f}^+ f^- \gamma^+ \gamma^-) = -2e^2 \left(\frac{\langle 24\rangle [q_4| \overset{①}{(|1]\langle 1| + |3]\langle 3|)} |q_3\rangle [31]}{\langle 13\rangle [13] \langle q_3 3\rangle [q_4 4]} + \frac{\langle 2 q_3\rangle [3| \overset{②}{(|1]\langle 1| + \cancel{|4]\langle 4|})} |4\rangle [q_4 1]}{\langle 14\rangle [14] \langle q_3 3\rangle [q_4 4]} \right)$$

$$① = \frac{\langle 24\rangle [q_4 1] \langle 1 q_3\rangle [31] + \langle 24\rangle [q_4 3] \langle 3 q_3\rangle [31]}{\langle 13\rangle [13] \langle q_3 3\rangle [q_4 4]}$$

$$= \frac{\langle 24\rangle [q_4 1] \langle q_3 1\rangle + \langle 24\rangle [q_4 3] \langle q_3 3\rangle}{\langle 13\rangle \langle q_3 3\rangle [q_4 4]} = \frac{\langle 24\rangle \langle q_3| (1+3) |q_4]}{\langle 13\rangle \langle q_3 3\rangle [q_4 4]}.$$

$$② = \frac{\langle 2 q_3\rangle [31] \langle 14\rangle [q_4 1]}{\langle 14\rangle [14] \langle q_3 3\rangle [q_4 4]} = \frac{\langle q_3 2\rangle [13] [q_4 1]}{[14] \langle q_3 3\rangle [q_4 4]}$$

$$①+② = \frac{\langle 24\rangle [14] [q_4 1] \langle q_3 1\rangle + \langle 24\rangle [14] [q_4 3] \langle q_3 3\rangle + \overbrace{\langle 13\rangle [13]}^{\langle 24\rangle [24]} \langle q_3 2\rangle [q_4 1]}{\underbrace{\langle 13\rangle [14]}_{-\langle 23\rangle [24]} \langle q_3 3\rangle [q_4 4]}$$

$$[q_4 1] \langle q_3 1\rangle + [q_4 3] \langle q_3 3\rangle = \langle q_3| (1+3) |q_4] = -\langle q_3| (2+4) |q_4]$$

$$= -\langle q_3 2\rangle [q_4 2] - \langle q_3 4\rangle [q_4 4]$$

STUCK :(

2.13) Via crossing symmetry, Compton scattering can be regarded as electron-positron annihilation, $e^- e^+ \to \gamma\gamma$.

Show that the differential cross section is $\frac{d\sigma}{d\Omega} = \alpha^2 \frac{1+\cos^2\theta}{1-\cos^2\theta} + \mathcal{O}(\alpha^4)$ in the limit $E_{cm} \gg m_e c^2$, where $\alpha = e^2/4\pi$.

$$\mathcal{A}_4(\bar{f}^+ f^- \gamma^+ \gamma^-) = 2e^2 \frac{\langle 24\rangle^2}{\langle 13\rangle\langle 23\rangle} \qquad |\mathcal{A}_4(+-+-)|^2 = (2e^2)^2 \frac{(\langle 24\rangle[24])^2}{\langle 13\rangle[13]\langle 23\rangle[23]}$$

$$\mathcal{A}_4(\bar{f}^+ f^- \gamma^- \gamma^+) = 2e^2 (3 \leftrightarrow 4) \qquad | \quad |^2 = (2e^2)^2 \frac{(\langle 23\rangle[23])^2}{\langle 14\rangle[14]\langle 24\rangle[24]}$$

$$\mathcal{A}_4(\bar{f}^- f^+ \gamma^+ \gamma^-) = 2e^2 [2| \frac{(|4\rangle[q_4| + |q_4]\langle 4|)}{[q_4 4]} \frac{(1+3)}{\langle 13\rangle[13]} \frac{(|3]\langle q_3| + |q_3\rangle[3|)}{\langle q_3 3\rangle} |1\rangle$$

$$+-+-: \quad 2e^2 \frac{\langle 2|(|4\rangle[q_4| + \cancel{|q_4]\langle 4|}^{\,0})(1+3)(\cancel{|3]\langle q_3|}^{\,0} + |q_3\rangle[3|)|1]}{\langle 13\rangle[13]\langle q_3 3\rangle[q_4 4]}$$

$$\shortparallel$$

$$\frac{\langle 24\rangle[q_4|(-|1]\langle 1| - |3]\langle 3|)|q_3\rangle[31]}{(\cdots)} = \frac{-\langle 24\rangle[q_4 1]\langle 1 q_3\rangle[31] - \langle 24\rangle[q_4 3]\langle 3 q_3\rangle[31]}{(\cdots)}$$

$$+ 2e^2 \frac{\langle 2|(\cancel{|3]\langle q_3|}^{\,0} + |q_3\rangle[3|)(1+4)(|4\rangle[q_4| + \cancel{|q_4]\langle 4|}^{\,0})|1]}{\langle 14\rangle[14]\langle q_3 3\rangle[q_4 4]}$$

$$\shortparallel$$

$$2e^2 \frac{\langle 2 q_3\rangle[3|1|4\rangle[q_4 1]}{(\cdots)} = 2e^2 \frac{\langle 2 q_3\rangle[31]\langle 41\rangle[q_4 1]}{\langle 14\rangle[14]\langle q_3 3\rangle[q_4 4]}$$

First deal w/ $\dfrac{[q_4 1]\langle 1 q_3\rangle}{\langle q_3 3\rangle[q_4 4]}$

$$\frac{\langle q_3 1\rangle}{\langle q_3 3\rangle} \cdot \frac{[13]}{[13]} = \frac{+\langle q_3|1|3]}{\langle q_3|3|1]}$$

$$\frac{\langle q_3 1\rangle}{\langle q_3 3\rangle} \cdot \frac{\langle 34\rangle}{\langle 34\rangle} = \frac{-\langle q_3 3\rangle\langle 41\rangle - \langle q_3 4\rangle\langle 13\rangle}{\langle q_3 3\rangle\langle 34\rangle}$$

$$\frac{\langle q_3 1\rangle}{\langle q_3 3\rangle} \cdot \frac{[13]}{[13]} = \frac{\langle q_3|\gamma^\mu|1]\langle 1|\gamma_\mu|3]}{\langle q_3|\gamma^\nu|1]\langle 3|\gamma_\nu|3]}$$

$$\langle ri\rangle\langle jk\rangle = -\langle rj\rangle\langle ki\rangle - \langle rk\rangle\langle ij\rangle$$

$$2 q_3 \cdot p_3 = (q_3 + p_3)^2 = \langle q_3 3\rangle[q_3 3]$$

$$\frac{[q_4 1]}{[q_4 4]} \frac{[34]}{[34]} =$$

anyways...

$$(\bar{f}^{1-} f^{2+} \gamma^{3-} \gamma^{4+})^* = \gamma^{4+} \gamma^{3-} \bar{f}^{2+} f^{1-} = f^- \bar{f}^+ \gamma^- \gamma^+$$

$$A_4(\bar{f}^- f^+ \gamma^- \gamma^+) = \left(A_4^*(\bar{f}^+ f^- \gamma^- \gamma^+)\right)_{1\leftrightarrow 2} = 2e^2 \left(\frac{\langle 24\rangle^2}{\langle 13\rangle\langle 23\rangle}\right)^*_{1\leftrightarrow 2} = 2e^2 \left(\frac{[24]^2}{[13][23]}\right)_{1\leftrightarrow 2}$$

$$= 2e^2 \frac{[14]^2}{[13][23]}$$

$$A_4(\bar{f}^- f^+ \gamma^+ \gamma^-) = 2e^2 \frac{[13]^2}{[14][24]}$$

$$| \quad |^2 = (2e^2)^2 \frac{(\langle 13\rangle[13])^2}{\langle 14\rangle[14]\langle 24\rangle[24]}$$

$$\sum |A_4|^2 = (2e^2)^2 \left[\frac{(\langle 24\rangle[24])^2}{\langle 13\rangle[13]\langle 23\rangle[23]} + \frac{(\langle 23\rangle[23])^2}{\langle 14\rangle[14]\langle 24\rangle[24]} + \frac{(\langle 14\rangle[14])^2}{\langle 13\rangle[13]\langle 23\rangle[23]} + \frac{(\langle 13\rangle[13])^2}{\langle 14\rangle[14]\langle 24\rangle[24]}\right]$$

$$= 2(2e^2)^2 \frac{(\langle 14\rangle[14])^2 + (\langle 24\rangle[24])^2}{\langle 14\rangle[14]\langle 24\rangle[24]}$$

$$E_{cm} = \frac{E_1 + E_2}{2}$$

$$= 2(2e^2)^2 \frac{(p_1 \cdot p_4)^2 + (p_2 \cdot p_4)^2}{p_1 \cdot p_4 \; p_2 \cdot p_4} = 2(2e^2)^2 \frac{(E_1 E_4 (\cos\theta - 1)^2 + E_2 E_4 (\cos\theta + 1)^2)}{E_1 E_2 E_4^2 (\cos\theta - 1)(-\cos\theta - 1)}$$

$$= \frac{2^4 e^4}{E^2} \frac{\cos^2\theta + 1}{\cos^2\theta - 1} = \frac{2^6}{E_{cm}^2} \frac{1 + \cos^2\theta}{1 - \cos^2\theta}$$

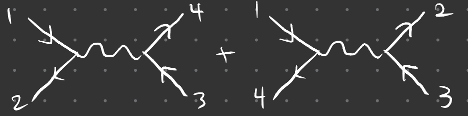
$$\cos^2\theta - 2\cos\theta + 1 + \cos^2\theta + 2\cos\theta + 1$$

$$= \frac{2 + 2\cos^2\theta}{-1 + \cos^2\theta}$$

n set $E_1 = E_2$

$$E_{cm}^2 = 4E^2$$

2.14) Calculate the tree-level Bhabha scattering process $e^-e^+ \to e^-e^+$,



$$iA_4(+-+-) = \bar{u}_{1+}\, ie\gamma^\mu v_{2-} \frac{-i\eta_{\mu\nu}}{(p_1+p_2)^2} \bar{u}_{3+}\, ie\gamma^\nu v_{4-} + (1\leftrightarrow 3)$$

$$= ie^2 \frac{[1|\gamma^\mu|2\rangle[3|\gamma_\mu|4\rangle}{\langle 12\rangle[12]} + (1\leftrightarrow 3) = 2ie^2\left(\frac{\langle 24\rangle[13]}{\langle 12\rangle[12]} + \frac{\langle 24\rangle[13]}{\langle 23\rangle[23]}\right)$$

$$= 2ie^2\langle 24\rangle[13]\left(\frac{1}{\langle 12\rangle[12]} + \frac{1}{\langle 23\rangle[23]}\right)$$

$$iA_4(+--+) = ie^2 \frac{[1|\gamma^\mu|2\rangle\langle 3|\gamma_\mu|4]}{\langle 12\rangle[12]} + (1\leftrightarrow 3)$$

$$= 2ie^2\left(\frac{\langle 23\rangle[14]}{\langle 12\rangle[12]} - \frac{\langle 12\rangle[34]}{\langle 23\rangle[23]}\right)$$

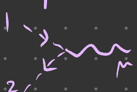
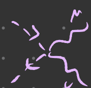
$$iA_4(-++-) = 2ie^2\left(\frac{\langle 14\rangle[23]}{\langle 12\rangle[12]} - \frac{\langle 34\rangle[12]}{\langle 23\rangle[23]}\right)$$

$$iA_4(-+-+) = 2ie^2\langle 13\rangle[24]\left(\frac{1}{\langle 12\rangle[12]} + \frac{1}{\langle 23\rangle[23]}\right)$$

$$\langle|A_4(f\bar{f}f\bar{f})|^2\rangle = 8e^4\left[\left(\langle 13\rangle[13]\left(\frac{1}{\langle 12\rangle[12]} + \frac{1}{\langle 23\rangle[23]}\right)\right)^2\right.$$

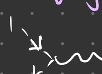
$$\left. + \left(\frac{\langle 23\rangle[14]}{\langle 12\rangle[12]} - \frac{\langle 12\rangle[34]}{\langle 23\rangle[23]}\right)\left(\frac{\langle 14\rangle[23]}{\langle 12\rangle[12]} - \frac{\langle 34\rangle[12]}{\langle 23\rangle[23]}\right)\right]$$

2.15) Consider massless scalar QED, w/ Lagrangian $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - |D\phi|^2 - \frac{1}{4}\lambda|\phi|^4$ (ϕ is a complex scalar).

vertices:  $= ie(p_2-p_1)^\mu$,  $= -2ie^2\eta_{\mu\nu}$,  $= -i\lambda$

Calculate the 3-particle amplitude $A_3(\phi\phi^*\gamma^-)$. Show that it is independent of reference spinors & write the result in a form that only involves angle brackets.

Use complex conjugation to write down $A_3(\phi\phi^*\gamma^+)$.

$iA_3(\phi\phi^*\gamma^-) =$  $= ie(p_2-p_1)_\mu\epsilon^\mu_-(p_3;q) = -ie(p_2-p_1)_\mu \frac{\langle 3|\gamma^\mu|q]}{\sqrt{2}[q3]}$

$$= \frac{-ie}{\sqrt{2}}\frac{\langle 3|(2-1)|q]}{[q3]} = \frac{-ie}{\sqrt{2}}\, 2\,\frac{\langle 3|2|q]}{[q3]} = \sqrt{2}\, ie\, \frac{\langle 32\rangle[2q]}{[q3]}\cdot\frac{\langle 12\rangle}{\langle 12\rangle}$$

$$= \sqrt{2}\, ie\, \frac{\langle 32\rangle\langle 13\rangle[q3]}{\langle 12\rangle[q3]} = \sqrt{2}\, ie\, \frac{\langle 13\rangle\langle 32\rangle}{\langle 12\rangle}$$

$$\Rightarrow A_3(\phi\phi^*\gamma^-) = \sqrt{2}\, e\, \frac{\langle 13\rangle\langle 32\rangle}{\langle 12\rangle}$$

$$A_3(\phi\phi^*\gamma^+) = A_3^*(\phi\phi^*\gamma^+) = \sqrt{2}\, e\, \frac{[13][32]}{[12]}$$

2.16) Consider the amplitude $A_4(\phi\phi^*\gamma\gamma)$. Show that for any choice of photon helicities, one can always pick the reference spinors s.t. the scalar-scalar-photon-photon contact term gives a vanishing contribution to the on-shell 4-pt amplitude.

$\sim \eta_{\mu\nu}\epsilon_3^\mu \epsilon_4^\nu$

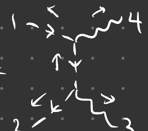
$$\gamma^+\gamma^+: \frac{1}{2}\frac{\langle q_3|\gamma^\mu|3]\langle q_4|\gamma_\mu|4]}{\langle q_3 3\rangle\langle q_4 4\rangle} = \frac{\langle q_3 q_4\rangle[34]}{\langle q_3 3\rangle\langle q_4 4\rangle} = 0 \text{ if } q_3 = q_4 = 1$$

same for $\gamma^-\gamma^-$ b/c it's the complex conj.

$$\gamma^+\gamma^-: \frac{1}{2}\frac{\langle q_3|\gamma^\mu|3][q_4|\gamma_\mu|4\rangle}{\langle q_3 3\rangle[q_4 4]} = \frac{\langle q_3 4\rangle[3 q_4]}{\langle q_3 3\rangle[q_4 4]} = 0 \text{ if } q_3 = 4,\ q_4 = 3$$

same for $\gamma^-\gamma^+$.

2.17) Calculate $A_4(\phi\phi^*\gamma\gamma)$ & massage the answer into a form that depends only on angle (or square) brackets & is manifestly independent of reference spinors.

$iA_4(\phi\phi^*\gamma^+\gamma^+) =$ 

$$+\,(3\leftrightarrow 4) = (ie)^2(-p_1-p_4-p_1)_\mu \epsilon_{4+}^\mu (p_2+p_2+p_3)_\nu \epsilon_{3+}^\nu + (3\leftrightarrow 4)$$

$$= \frac{e^2}{2}\frac{2\langle q_4|1|4]\,2\langle q_3|2|3]}{\langle q_4 4\rangle\langle q_3 3\rangle} + (3\leftrightarrow 4) = 2e^2\frac{\langle q_4 1\rangle[14]\langle q_3 2\rangle[23]}{\langle q_4 4\rangle\langle q_3 3\rangle} + (3\leftrightarrow 4)$$

$\langle q_3 1\rangle[13]\langle q_4 2\rangle[24]$

$$\frac{\langle q_4 1\rangle}{\langle q_4 4\rangle}\frac{\langle 24\rangle}{\langle 24\rangle} = \frac{-\langle q_4 2\rangle\langle 41\rangle - \langle q_4 4\rangle\langle 12\rangle}{\langle q_4 4\rangle\langle 24\rangle}$$

$$\frac{\langle q_4 2\rangle}{\langle q_4 4\rangle}$$

answer: $q_4 = 3,\ q_3 = 4$

$$-2e^2\,\frac{\langle 31\rangle[14]\langle 42\rangle[23]}{\langle 34\rangle^2}$$

still need get rid of square/angle brackets

$$\frac{\langle q_4 1\rangle[14]\langle q_3 2\rangle[23]}{\langle q_3 3\rangle} + \frac{\langle q_3 1\rangle[13]\langle q_4 2\rangle[24]}{\langle q_3 3\rangle}$$

$$||$$

$$[14][23] = -[12][34] + [13][24]$$

$$\langle q_4 1\rangle\langle q_3 2\rangle\big([13][24] - [12][34]\big) + \langle q_3 1\rangle\langle q_4 2\rangle\big([13][24]\big)$$

$$\langle q_3 3\rangle\langle q_4 4\rangle = -\langle q_3 q_4\rangle\langle 43\rangle - \langle q_3 4\rangle\langle 3 q_4\rangle$$

$$\frac{\langle q_4 1\rangle}{\langle q_4 4\rangle}\cdot\frac{\langle 34\rangle}{\langle 34\rangle} = \frac{\langle q_4 3\rangle\langle 14\rangle + \langle q_4 4\rangle\langle 31\rangle}{\langle q_4 4\rangle\langle 34\rangle}$$

$$= \frac{\langle q_4 2\rangle\langle 14\rangle + \langle q_4 4\rangle\langle 21\rangle}{\langle q_4 4\rangle\langle 24\rangle}$$

$$\langle q_4 3\rangle\langle 14\rangle\langle 24\rangle + \langle q_4 4\rangle\langle 31\rangle\langle 24\rangle = \langle q_4 2\rangle\langle 14\rangle\langle 34\rangle + \langle q_4 4\rangle\langle 21\rangle\langle 34\rangle$$

$$= \langle q_4 4\rangle\big(-\langle 32\rangle\langle 41\rangle - \langle 34\rangle\langle 12\rangle\big)$$

$$= -\langle q_4 4\rangle\langle 23\rangle\langle 14\rangle + \langle q_4 4\rangle\langle 21\rangle\langle 34\rangle$$

$$\langle q_4 3\rangle\langle 14\rangle\langle 24\rangle - \langle q_4 4\rangle\langle 23\rangle\langle 14\rangle = \langle q_4 2\rangle\langle 14\rangle\langle 34\rangle$$

STUCK :(

2.19) Show that $A_4(\phi\phi^*\phi\phi^*) = -\lambda + 2e^2\left(1+\frac{\langle13\rangle^2\langle24\rangle^2}{\langle12\rangle\langle23\rangle\langle34\rangle\langle41\rangle}\right)$

$iA_4(\phi\phi^*\phi\phi^*) =$

$$= -i\lambda + (ie)^2(p_2-p_1)^\mu \frac{-i\eta_{\mu\nu}}{(p_1+p_2)^2}(p_4-p_3)^\nu + (1\leftrightarrow4)$$

$$= -i\lambda + ie^2\frac{p_2\cdot p_4+p_1\cdot p_3-p_2\cdot p_3-p_1\cdot p_4}{(p_1+p_2)^2} + (1\leftrightarrow4)$$

$$= -i\lambda + ie^2\left(\frac{\langle24\rangle[24]-\langle23\rangle[23]}{\langle12\rangle[12]} + \frac{\langle12\rangle[12]-\langle24\rangle[24]}{\langle24\rangle[24]}\right)$$

$$= -i\lambda+ie^2\left(\frac{\langle24\rangle^2[24]^2-\langle23\rangle[23]\langle24\rangle[24]+\langle12\rangle^2[12]^2-\langle12\rangle[12]\langle24\rangle[24]}{\langle12\rangle[12]\langle24\rangle[24]}\right)$$

$$=-i\lambda+2ie^2\left(1+\frac{\langle12\rangle^2[12]^2+\langle24\rangle^2[24]^2}{\langle12\rangle[12]\langle24\rangle[24]}\right)$$

$\langle1|2|4]\langle4|2|1]$

$\langle12\rangle[12]=\langle34\rangle[34]$ $\quad \langle12\rangle[24] = -\langle1|2|4]$

$\langle24\rangle[12]=\langle42\rangle[21] = -\langle4|2|1]$

$\frac{[12]}{[24]}=$ $\quad = 2+\frac{2s^2+2t^2}{st}$

$\langle12\rangle[12]=\langle1|2|1]$

$\langle1|2|1]^2+\langle4|2|4]^2$

$s=\langle12\rangle[12]$

$t=\langle13\rangle[13]=\langle24\rangle[24]$

$u=\langle23\rangle[23]=\langle14\rangle[24]$

$\frac{s^2+t^2}{}$

$\frac{\langle12\rangle[12]}{\langle24\rangle[24]}+\frac{\langle24\rangle[24]}{\langle12\rangle[12]}$

$s+t+u=0$

$\langle23\rangle[23]=-\langle12\rangle[12]-\langle24\rangle[24]$

$s^2+t^2=(p_1+p_2)^4+(p_2+p_4)^4$ $\quad \frac{s^2+t^2}{st}=\frac{u^2}{st}+2$

$-\frac{\langle2|4|1]}{\langle4|2|1]} = \frac{[14]}{[12]}\cdot\frac{\langle24\rangle}{\langle24\rangle}$ $\quad =(s+t)^2-2st$

$=(-s-t)^2-2st = u^2-2st$

$\frac{\langle14\rangle^2[14]^2}{\langle12\rangle[12]\langle24\rangle[24]}$ $\quad 1+\frac{s^2+t^2}{st}=1+\frac{u^2-2st}{st}$

$=-1+\frac{u^2}{st}$

2.20) Calculate the gluon partial amplitude $\mathcal{A}_3[1^+2^+3^-]$.

$$\mathcal{A}_3[++-] = -\sqrt{2}\left[(\epsilon_{1+}\cdot\epsilon_{2+})(\epsilon_{3-}\cdot p_1) + (\epsilon_{2+}\cdot\epsilon_{3-})(\epsilon_{1+}\cdot p_2) + (\epsilon_{3-}\cdot\epsilon_{1+})(\epsilon_{2+}\cdot p_3)\right]$$

$$= \frac{1}{2}\Big(\langle q_1|\gamma^\mu|1]\langle q_2|\gamma_\mu|2]\langle 3|1|q_3] + \langle q_2|\gamma^\mu|2]\langle 3|\gamma_\mu|q_3]\langle q_1|2|1]$$
$$+ \langle 3|\gamma^\mu|q_3]\langle q_1|\gamma_\mu|1]\langle q_2|3|2]\Big)\cdot\frac{1}{\langle q_1 1\rangle\langle q_2 2\rangle[q_3 3]}$$

$$= \frac{\langle q_1 q_2\rangle[12]\langle 31\rangle[q_3 1] + \langle q_2 3\rangle[2 q_3]\langle q_1 2\rangle[12] + \langle 3 q_1\rangle[q_3 1]\langle q_2 3\rangle[23]}{\langle q_1 1\rangle\langle q_2 2\rangle[q_3 3]}$$

this is 0 if $|1] \propto |2] \propto |3]$, so assume $|1\rangle \propto |2\rangle \propto |3\rangle$

Also, $\langle q_1 2\rangle[12] = \langle q_1|2|1] = -\langle q_1|3|1] = -\langle q_1 3\rangle[13]$

$$\Rightarrow A_3(++-) = \frac{\langle q_2 3\rangle\langle q_1 3\rangle\left([q_3 2][13] + [q_3 1][32]\right)}{\langle q_1 1\rangle\langle q_2 2\rangle[q_3 3]}$$

$$= \frac{\langle q_2 3\rangle\langle q_1 3\rangle\cancel{[q_3 3]}[12]}{\langle q_1 1\rangle\langle q_2 2\rangle\cancel{[q_3 3]}} = \frac{-\cancel{\langle q_1 1\rangle}[12]\cancel{\langle q_2 2\rangle}[12]}{\cancel{\langle q_1 1\rangle}\cancel{\langle q_2 2\rangle}[23][13]} = \frac{[12]^2}{[23][31]}$$

2.21) Calculate $\epsilon_h(p;q)\cdot\epsilon_{h'}(k;q')$ for all combos of h, h'.

Show $\epsilon_\pm(p;q)\cdot\epsilon_\pm(k,q') = 0$ if $q = q'$.

How can you make $\epsilon_\pm(p;q)\cdot\epsilon_\mp(k;q')$ vanish?

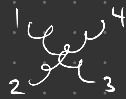
$$\epsilon_+(p;q)\cdot\epsilon_+(k;q') = \frac{\langle q|\gamma^\mu|p]\langle q'|\gamma_\mu|k]}{2\langle qp\rangle\langle q'k\rangle} = \frac{\langle qq'\rangle[pk]}{\langle qp\rangle\langle q'k\rangle} = 0 \text{ if } q = q'$$

$$\epsilon_+(p;q)\cdot\epsilon_-(k;q') = \frac{\langle q|\gamma^\mu|p]\langle k|\gamma_\mu|q']}{2\langle qp\rangle[q'k]} = \frac{\langle qk\rangle[pq']}{\langle qp\rangle[q'k]} = 0 \text{ if } q = k \text{ or } q' = p$$

$$\epsilon_-(p;q)\cdot\epsilon_+(k;q') = \frac{\langle pq'\rangle[qk]}{[qp]\langle q'k\rangle} = 0 \text{ if } q = k \text{ or } q' = p$$

$$\epsilon_-(p;q)\cdot\epsilon_+(k;q') = \frac{\langle pk\rangle[qq']}{[qp]\langle q'k\rangle} = 0 \text{ if } q = q'$$

2.22) Use the previous exercise to show that for any choice of gluon helicities, you can choose the polarization vectors s.t. the contribution from the 4-gluon contact term to the 4-gluon amplitude vanishes.



$= (\epsilon_1 \epsilon_3)(\epsilon_2 \epsilon_4)$

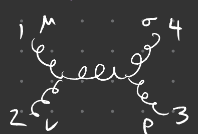
$$\left.\begin{array}{l} ++++ \\ +-+- \\ -+-+ \\ ---- \end{array}\right\} \quad \text{choose } q_1 = q_3 = 2, \; q_2 = q_4 = 1$$

$$\left.\begin{array}{l} +++- \\ ---+ \\ +-++ \\ -+-- \end{array}\right\} \quad \text{choose } q_1 = q_3 = 2, \; q_2 = 4, \; q_4 = 2$$

$$\left.\begin{array}{l} -+++ \\ +--- \\ ++-+ \\ +-++ \end{array}\right\} \quad \text{choose } q_1 = 3, \; q_3 = 1, \; q_2 = q_4 = 3$$

$$\left.\begin{array}{l} ++-- \\ --++ \\ +--+ \\ -++- \end{array}\right\} \quad \text{choose } q_1 = 3, \; q_3 = 1, \; q_2 = 4, \; q_4 = 2$$

2.23) Use a well-chosen set of reference spinors to show that the entire 4-gluon amplitude vanishes if all 4 gluons have the same helicity.

 $+$ 

$$\sim \epsilon_{1\mu}\epsilon_{2\nu}\epsilon_{3\rho}\epsilon_{4\sigma}\left(\eta^{\mu\nu}p_1^\alpha + \eta^{\nu\alpha}p_2^\mu + \eta^{\alpha\mu}(p_1+p_2)^\nu\right) \cdot \frac{\eta_{\alpha\beta}}{(p_1+p_2)^2} \cdot \left(\eta^{\rho\sigma}p_3^\beta + \eta^{\sigma\beta}p_4^\rho + \eta^{\beta\rho}(p_1+p_2)^\sigma\right) + (1 \leftrightarrow 3)$$

$$= \frac{1}{(p_1+p_2)^2}\Big[(\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot\epsilon_4)(p_1\cdot p_3) + (\epsilon_1\cdot\epsilon_2)(\epsilon_4\cdot p_1)(p_4\cdot\epsilon_3) + (\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot p_1)((p_1+p_2)\cdot\epsilon_4)$$
$$+ (\epsilon_2\cdot p_3)(\epsilon_1\cdot p_2)(\epsilon_3\cdot\epsilon_4) + (\epsilon_2\cdot\epsilon_4)(\epsilon_1\cdot p_2)(p_4\cdot\epsilon_3) + (\epsilon_2\cdot\epsilon_3)(\epsilon_1\cdot p_2)((p_1+p_2)\cdot\epsilon_3)$$
$$+ (\epsilon_1\cdot p_3)((p_1+p_2)\cdot\epsilon_2)(\epsilon_3\cdot\epsilon_4) + (\epsilon_1\cdot\epsilon_4)((p_1+p_2)\cdot\epsilon_2)(\epsilon_3\cdot p_4)$$
$$+ (\epsilon_1\cdot\epsilon_3)((p_1+p_2)\cdot p_2)((p_1+p_2)\cdot\epsilon_4)\Big]$$

choose $q_1 = q_2 = q_3 = q_4 = 1+2$ so all $\epsilon_i \cdot \epsilon_j = 0$

(same for $(1 \leftrightarrow 3)$ diagram)

2.24) Calculate the color-ordered 4-gluon tree amplitude $A_4(1^-2^-3^+4^+)$ using the color-ordered Feynman rules & a smart choice of reference spinors.

Show that the answer has the form

$$A_4(1^-2^-3^+4^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}.$$

Choose $q_1 = q_2 = 3 \Rightarrow \epsilon_1\cdot\epsilon_2 = 0,\ \epsilon_1\cdot\epsilon_3 = 0,\ \epsilon_2\cdot\epsilon_3 = 0$

$q_3 = q_4 = 2 \Rightarrow \epsilon_3\cdot\epsilon_4 = 0,\ \epsilon_2\cdot\epsilon_4 = 0,\ \epsilon_2\cdot\epsilon_3 = 0$

$\epsilon_1\cdot\epsilon_4 \neq 0$

$$\Rightarrow A_4(--++) = 2\,\frac{\epsilon_1^-\cdot\epsilon_4^+\,(p_1\cdot\epsilon_2^- + \cancelto{0}{p_2\cdot\epsilon_2^-})(\epsilon_3^+\cdot p_4)}{\langle 12\rangle[12]}$$

$$= \frac{1}{2}\,\frac{\langle 1|\gamma^\mu|3]\langle 2|\gamma_\mu|4]\langle 2|1|3]\langle 2|4|3]}{\langle 12\rangle[12][31]\langle 24\rangle[32]\langle 23\rangle} = \frac{\langle 12\rangle[34]\langle 21\rangle\cancel{[31]}\cancel{\langle 24\rangle}[34]}{\langle 12\rangle[12]\cancel{[31]}\cancel{\langle 24\rangle}[32]\langle 23\rangle}$$

$$= \frac{\langle 12\rangle^3[34]^2}{\langle 12\rangle\langle 34\rangle\cancel{[34]}\langle 23\rangle[23]}$$

$$= \frac{\langle 12\rangle^4[12]}{\langle 12\rangle\langle 23\rangle\langle 34\rangle^2[23]}$$

$$= \frac{\langle 24\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

$$\left(\begin{aligned} &\langle 14\rangle[12] \stackrel{?}{=} \langle 34\rangle[23] \\ &\langle 4|1|2] \stackrel{?}{=} -\langle 4|3|2] \\ &\langle 4|1|2] + \langle 4|3|2] \stackrel{?}{=} 0 \\ &\langle 4|1+2+3+4|2] = 0 \quad \smile \end{aligned}\right)$$

2.25) Show that $A_4(--++)$ can also be written as $A_4(--++) = \frac{[34]^4}{[12][23][34][41]}$.

Starting from ,

$$A_4(--++) = \frac{[34]^3\langle 12\rangle}{\langle 12\rangle\langle 23\rangle[12][23][34]} = \frac{[34]^4\langle 34\rangle}{[12]^2[23][34]\langle 23\rangle}$$

$$= -\frac{[34]^4}{[12][23][34][41]}$$

need to fix sign

$$\left(\begin{aligned} &\frac{\langle 34\rangle}{[12]\langle 23\rangle} \stackrel{?}{=} \frac{1}{[14]} \\ &\langle 34\rangle[14] \stackrel{?}{=} \langle 23\rangle[12] \\ &\langle 3|4|1] \stackrel{?}{=} -\langle 3|2|1] \end{aligned}\right)$$

2.26) Convince yourself that in general if all helicities are flipped $(h_i \mapsto -h_i)$ then the resulting amplitude $A_n[1^{h_1}\cdots n^{h_n}]$ is obtained from $A_n[1^{-h_1}\cdots n^{-h_n}]$ by exchanging all angle & square brackets. ✓

2.27) Show the U(1) decoupling identity

$$\underbrace{A_n[123\cdots n] + A_n[213\cdots n] + \cdots + A_n[23\cdots 1n]}_{n-1 \text{ terms}} = 0$$

holds for $n=4$, helicities $--++$.

$$\mathcal{A}_4[1^-2^-3^+4^+] + \mathcal{A}_4[2^-1^-3^+4^+] + \mathcal{A}_4[2^-3^+1^-4^+]$$

$$= \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} + \frac{\langle 21\rangle^4}{\langle 21\rangle\langle 13\rangle\langle 34\rangle\langle 42\rangle} + \frac{\langle 21\rangle^4}{\langle 23\rangle\langle 31\rangle\langle 14\rangle\langle 42\rangle}$$

$$= \langle 12\rangle^4\left(\frac{-1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 14\rangle} + \frac{1}{\langle 12\rangle\langle 13\rangle\langle 34\rangle\langle 24\rangle} + \frac{1}{\langle 23\rangle\langle 13\rangle\langle 14\rangle\langle 24\rangle}\right)$$

$$= \langle 12\rangle^4 \cdot \frac{-\langle 13\rangle\langle 24\rangle + \langle 14\rangle\langle 23\rangle + \langle 12\rangle\langle 34\rangle}{\langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle}$$

$$\sim \langle 12\rangle\langle 43\rangle + \langle 14\rangle\langle 32\rangle + \langle 14\rangle\langle 23\rangle + \langle 12\rangle\langle 34\rangle = 0$$

2.28) Show that for $n=4$, the Kleis-Kuijf relation is equivalent to the U(1) decoupling relation.

$$\mathcal{A}_4[1\{2\}4\{3\}] = -(\mathcal{A}_4[1234] + \mathcal{A}_4[1324])$$

$$\Rightarrow \mathcal{A}_4[1243] + \mathcal{A}_4[1234] + \mathcal{A}_4[1324] = 0$$

$$= \mathcal{A}_4[3421] + \mathcal{A}_4[4321] + \mathcal{A}_4[4231] = 0$$

this is the U(1) decoupling relation w/ $\begin{cases} 1 \to 3 \\ 2 \to 4 \\ 3 \to 2 \\ 4 \to 1 \end{cases}$

2.29) Start w/ $\mathcal{A}_5[1\{23\}5\{4\}]$ to show that the K-K relation gives

$\mathcal{A}_5[12345] + \mathcal{A}_5[12354] + \mathcal{A}_5[12435] + \mathcal{A}_5[14235] = 0.$

Then show that this combined w/ the U(1) decoupling gives

$\mathcal{A}_5[12534] = \mathcal{A}_5[12435] + \mathcal{A}_5[14235] + \mathcal{A}_5[14325]$.

$$\mathcal{A}_5[1\{23\}5\{4\}] = -(\mathcal{A}_5[12345] + \mathcal{A}_5[12435] + \mathcal{A}_5[14235])$$

$$\Rightarrow \mathcal{A}_5[12345] + \mathcal{A}_5[12354] + \mathcal{A}_5[12435] + \mathcal{A}_5[14235] = 0.$$

U(1) decoupling: $\mathcal{A}_5[54321] + \mathcal{A}_5[45321] + \mathcal{A}_5[43521] + \mathcal{A}_5[43251] = 0$

$$\Rightarrow \mathcal{A}_5[12534] = -(\mathcal{A}_5[12345] + \mathcal{A}_5[12354] + \mathcal{A}_5[15234])$$

$$\mathcal{A}_5[12354] + \mathcal{A}_5[12345] + \mathcal{A}_5[12435] + \mathcal{A}_5[14235] = 0$$

$$\mathcal{A}_5[12534] = -(-\cancel{\mathcal{A}_5[12354]} - \mathcal{A}_5[12435] - \mathcal{A}_5[14235]$$

$$+ \cancel{\mathcal{A}_5[12354]} + \mathcal{A}_5[15234])$$

$$= A_5[12435] + A_5[14235] + A_5[43251]$$

$$= A_5[12435] + A_5[14235] + A[14325]$$

2.30) Use the Parke-Taylor formula to verify

① $A_5[12345] + A_5[12354] + A_5[12435] + A_5[14235] = 0$

② $s_{14} A_4[1234] - s_{13} A_4[1243] = 0$

③ $s_{12} A_5[21345] - s_{23} A_5[13245] - (s_{23} + s_{24}) A_5[13425] = 0$

① Suppose 1, 2 have (-) helicity.

$$\langle 12\rangle^4 \left(\frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} + \frac{1}{\langle 12\rangle\langle 23\rangle\langle 35\rangle\langle 54\rangle\langle 41\rangle} + \frac{1}{\langle 12\rangle\langle 24\rangle\langle 43\rangle\langle 35\rangle\langle 51\rangle} + \frac{1}{\langle 14\rangle\langle 42\rangle\langle 23\rangle\langle 35\rangle\langle 51\rangle} \right)$$

$$= \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 14\rangle\langle 15\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle\langle 45\rangle} \Big(-\langle 14\rangle\langle 24\rangle\langle 35\rangle + \langle 15\rangle\langle 24\rangle\langle 34\rangle + \langle 14\rangle\langle 23\rangle\langle 45\rangle + \langle 12\rangle\langle 34\rangle\langle 45\rangle \Big)$$

$$\sim -\langle 14\rangle\langle 24\rangle\langle 35\rangle + \langle 15\rangle\langle 24\rangle\langle 34\rangle + \langle 13\rangle\langle 24\rangle\langle 45\rangle$$

$$= \langle 24\rangle\big(\langle 14\rangle\langle 53\rangle + \langle 15\rangle\langle 34\rangle + \langle 13\rangle\langle 45\rangle \big) = 0$$

② Again assume $1^- 2^-$.

$$\langle 12\rangle^4 \left(\frac{\langle 14\rangle[14]}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} - \frac{\langle 13\rangle[13]}{\langle 12\rangle\langle 24\rangle\langle 43\rangle\langle 31\rangle} \right)$$

$$\sim \frac{\langle 14\rangle[14]}{\langle 23\rangle\langle 41\rangle} + \frac{\langle 13\rangle[13]}{\langle 24\rangle\langle 31\rangle} = -\frac{[14]}{\langle 23\rangle} - \frac{[13]}{\langle 24\rangle} = -\frac{\langle 24\rangle[14] + \langle 23\rangle[13]}{\langle 23\rangle\langle 24\rangle}$$

$$\sim \langle 2|(4+3)|1] = -\langle 2|(1+2)|1] = 0$$

③ Again assume $1^- 2^-$.

$$\langle 12\rangle^4 \left(\frac{\langle 12\rangle[12]}{\langle 21\rangle\langle 13\rangle\langle 34\rangle\langle 45\rangle\langle 52\rangle} - \frac{\langle 23\rangle[23]}{\langle 13\rangle\langle 32\rangle\langle 24\rangle\langle 45\rangle\langle 51\rangle} - \frac{\langle 23\rangle[23] + \langle 24\rangle[24]}{\langle 13\rangle\langle 34\rangle\langle 42\rangle\langle 25\rangle\langle 51\rangle} \right)$$

$$\sim \frac{[12]}{\langle 13\rangle\langle 25\rangle\langle 34\rangle\langle 45\rangle} - \frac{[23]}{\langle 13\rangle\langle 15\rangle\langle 24\rangle\langle 45\rangle} - \frac{\langle 23\rangle[23] + \langle 24\rangle[24]}{\langle 13\rangle\langle 15\rangle\langle 24\rangle\langle 25\rangle\langle 34\rangle}$$

$$\sim [12]\langle 24\rangle\langle 15\rangle - [23]\langle 25\rangle\langle 34\rangle - [23]\langle 23\rangle\langle 45\rangle - [24]\langle 24\rangle\langle 45\rangle$$

$$= [12]\langle 24\rangle\langle 15\rangle - [23]\langle 24\rangle\langle 35\rangle - [24]\langle 24\rangle\langle 45\rangle$$

$\sim \langle 15\rangle[12] - \langle 35\rangle[23] - \langle 45\rangle[24] = \langle 5|(1+3+4)|2] = -\langle 5|(2+5)|2] = 0$

2.31) Suppose we use the color basis

$$\left\{ c_s = \tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4},\quad c_t = \tilde{f}^{a_1 a_3 b}\tilde{f}^{b a_4 a_2},\quad c_u = \tilde{f}^{a_1 a_4 b}\tilde{f}^{b a_2 a_3} \right\}$$

to write the full 4-pt gluon amplitude as

$A_4^{\text{full, tree}} = \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}$ for some numerator factors n_i that depend on the kinematic variables & polarizations.

Write each c_i in terms of the traces $\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$, $\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3})$, & $\mathrm{Tr}(T^{a_1}T^{a_3}T^{a_2}T^{a_4})$.

Check that the Jacobi identity $c_s + c_t + c_u = 0$ holds. Then convert $A_4^{\text{full, tree}}$ into a basis w/ those 3 traces.

$$c_s = (-\mathrm{Tr}(12b) + \mathrm{Tr}(21b))(-\mathrm{Tr}(b34) + \mathrm{Tr}(3b4))$$

$$= \mathrm{Tr}(12b)\mathrm{Tr}(b34) - \mathrm{Tr}(12b)\mathrm{Tr}(3b4) - \mathrm{Tr}(21b)\mathrm{Tr}(b34) + \mathrm{Tr}(21b)\mathrm{Tr}(3b4)$$

$$= 1234 - \cancel{\tfrac{1}{N}\,1234} - 1243 + \cancel{\tfrac{1}{N}\,1243} - 2134 + \cancel{\tfrac{1}{N}\,2134} + 2143 - \cancel{\tfrac{1}{N}\,2143}$$

$$= 1234 - 1243 - 1243 + 1234$$

$$= 2\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) - 2\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3})$$

$$c_t = (-13b + 31b)(-b42 + 4b2) = 1342 - 1324 - 3142 + 3124$$

$$= 1243 - 1324 - 1324 + 1243$$

$$= 2\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3}) - 2\mathrm{Tr}(T^{a_1}T^{a_3}T^{a_2}T^{a_4})$$

$$c_u = (-14b + 41b)(-b23 + 2b3) = 1423 - 1432 - 4123 + 4132$$

$$= 2\mathrm{Tr}(T^{a_1}T^{a_3}T^{a_2}T^{a_4}) - 2\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$$

$$c_s + c_t + c_u \sim 1234 - 1243 + 1243 - 1324 + 1324 - 1234 = 0.$$

$$A_4^{\text{full, tree}} = \frac{2n_s}{s}(1234 - 1243) + \frac{2n_t}{t}(1243 - 1324) + \frac{2n_u}{u}(1324 - 1234)$$

$$= 1234\left(\frac{2n_s}{s} - \frac{2n_u}{u}\right) + 1243\left(\frac{2n_t}{t} - \frac{2n_s}{s}\right) + 1324\left(\frac{2n_u}{u} - \frac{2n_t}{t}\right)$$

Now use the cyclic & reflection properties of the trace & color-ordered to write the full amplitude $A_4^{\text{full, tree}} = g^2\left(A_4[1234]\, 1234 + \sigma(234)\right)$ in terms of the traces w/ the same 3 orderings 1234, 1243, 1324.

$$A_4^{\text{full, tree}} = g^2\left(\text{"}1234\text{"} + \text{"}1243\text{"} + \text{"}1342\text{"} + \text{"}1324\text{"} + \text{"}1423\text{"} + \text{"}1432\text{"}\right)$$

$$= 2g^2\left(A_4[1234]\, 1234 + A_4[1243]\, 1243 + A_4(1324)\, 1324\right)$$

Compare the two expressions for $A_4^{\text{full, tree}}$ to read off the relationship b/w the numerator factors n_i & the color-ordered amplitudes.

Show that it follows directly from these expressions that the color-ordered amplitudes satisfy the $n=4$ $U(1)$ decoupling identity.

$$A_4[1234] = \frac{1}{g^2}\left(\frac{n_s}{s} - \frac{n_u}{u}\right), \quad A_4[1243] = \frac{1}{g^2}\left(\frac{n_t}{t} - \frac{n_s}{s}\right), \quad A_4[1324] = \frac{1}{g^2}\left(\frac{n_u}{u} - \frac{n_t}{t}\right)$$

$$A_4[1234] + A_4[2134] + A_4[2314] = A_4[1234] + A_4[1243] + A_4[1324] = 0$$

Note that the numerator factors are not unique. Suppose $\exists$ choice of n_i s.t. $n_s + n_t + n_u = 0$. Show that this implies the color-ordered amplitudes satisfy the BCJ relation $s_{14}A_4[1234] - s_{13}A_4[1243] = 0$.

$$s_{14}A_4[1234] - s_{13}A_4[1243] \sim u\left(\frac{n_s}{s} - \frac{n_u}{u}\right) - t\left(\frac{n_t}{t} - \frac{n_s}{s}\right) = -n_t - n_u + \frac{u n_s - t n_s}{s}$$

$$\sim -sn_t - sn_u + (u-t)n_s = -s(n_t + n_u + n_s) = 0$$

2.32) Write down spinor helicity formulas for the possible color-ordered 3-pt amplitudes w/ 2 gluinos (massless spin-½ adjoints) & 1 gluon.

Work backwards to find out what the interaction term looks like in the Lagrangian.

Assume only ⟨⟩ brackets, so

$$A_3(\psi^{h_1}\psi^{h_2}g^{h_3}) = c\langle 12\rangle^{h_3-h_1-h_2}\langle 13\rangle^{h_2-h_1-h_3}\langle 23\rangle^{h_1-h_2-h_3}$$

$$A_3(+-+) = c\langle 12\rangle\langle 13\rangle^{-2}\langle 23\rangle^{0} = c\frac{\langle 12\rangle}{\langle 13\rangle^2} \sim [m]^{-1} \Rightarrow [c] = [m]^2$$

If instead only [] brackets, $A_3(+-+) = \tilde{c}\,\frac{[13]^2}{[12]} \sim [m]^1 \Rightarrow [\tilde{c}] = 1$

$[\psi] = [m]^{3/2}$, $[A] = [m]$, $[\partial] = [m]$ $\Rightarrow$ can have $\tilde{c}\psi\psi A$

Due to locality, can't have $1/\partial$ terms, so choose [ ] version.

$\mathcal{A}_3(+--) = c[12][13]^0[23]^{-2} = \tilde{c}\frac{[12]}{[23]^2} \sim [m]^{-1}$ $\Rightarrow [\tilde{c}] = [m]^2$

$\Rightarrow$ choose $\langle\rangle$ version, $\mathcal{A}_3(+--) = c\frac{\langle 23\rangle^2}{\langle 12\rangle}$ $\Rightarrow [c] = 1$

$\Rightarrow$ still only $c\psi\psi A$ dunno if I did this right!

2.33) Suppose someone gives you the amplitudes (a), (b), (c) for massless particles w/ all momenta outgoing.

What are the helicities?

What is $[g_i]$?

In each case, try to figure out which theory could produce such an amplitude.

a) $\mathcal{A}_5 = g_a \frac{[13]^4}{[12][23][34][45][51]}$

Helicities: $1^+2^-3^+4^-5^-$

$\mathcal{A}_5 \sim [m]^{-1}$, needs to be $[m]^{-1} \Rightarrow [g_a] = 1$

$\mathcal{L}$ can have $g_a AAAA\frac{1}{\partial}A$

b) $\mathcal{A}_4 = g_b \frac{\langle 14\rangle\langle 24\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle}$

$-2h_1 = -1+1 = 0$

$-2h_2 = -1-1+2 = 0$

$-2h_3 = -1-1 = -2$

$-2h_4 = 1+3-1 = 3$

helicities: $h_1 = h_2 = 0$, $h_3 = 1$, $h_4 = -3/2$

$\mathcal{A}_4 \sim 1$, needs to be $1 \Rightarrow [g_b] = 1$

$[\phi] = [m]$, $[\psi] = [m]^{3/2}$ $\Rightarrow \mathcal{L}$ can have $\phi\phi A\psi$ (?)

c) $\mathcal{A}_4 = g_c \frac{\langle 12\rangle^7[12]}{\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle}$

$-2h_1 = 7-1-1-1 = 4$

$-2h_2 = 7-1-1-1 = 4$

$-2h_3 = -1-1-1 = -3$

$-2h_4 = -1-1-1 = -3$

helicities: $h_1 = h_2 = -2$, $h_3 = h_4 = 3/2$

$A_4 \sim [m]^3$, needs to be $1 \Rightarrow [g_c] = [m]^{-2}$

$\mathcal{L}$ can have $g_c^2(\partial A)^2(\partial\psi)\psi$

$g_c^2 A(\partial A)(\partial\psi)^2$

2.34) Use the little group scaling to write down on-shell 3-pt amplitudes for graviton scattering. Check the mass dimensions & compare w/ the 3-pt gluon amplitudes.

$$A_3(h_1 h_2 h_3) = c\langle 12\rangle^{h_3-h_1-h_2}\langle 13\rangle^{h_2-h_1-h_3}\langle 23\rangle^{h_1-h_2-h_3}$$

$$A_3(+++) = \frac{c_0}{(\langle 12\rangle\langle 23\rangle\langle 31\rangle)^2} \quad \text{or} \quad \tilde c_0([12][23][31])^2 \quad \sim [m]^6[\tilde c_0] \qquad [\tilde c_0] = [m]^{-5}$$

$$A_3(++-) = c_1\frac{\langle 13\rangle^2\langle 23\rangle^2}{\langle 12\rangle^6} \quad \text{or} \quad \tilde c_1\frac{[12]^6}{([13][23])^2} \quad \sim [m]^2[\tilde c_1] \qquad [\tilde c_1] = [m]^{-1}$$

$$A_3(+--) = c_2\frac{\langle 23\rangle^6}{\langle 12\rangle^2\langle 13\rangle^2} \quad \text{or} \quad \tilde c_2\frac{([12][13])^2}{[23]^6} \qquad [c_2] = [m]^{-1}$$

$$A_3(---) = c_3(\langle 12\rangle\langle 13\rangle\langle 23\rangle)^2 \quad \text{or} \quad \frac{\tilde c_3}{([12][13][23])^2} \qquad [c_3] = [m]^{-5}$$

for gluons,

$$A_3(+++) = \tilde c_0[12][13][23] \sim [m]^3 \qquad [\tilde c_0] = [m]^{-2}$$

$$A_3(++-) = \tilde c_1\frac{[12]^3}{[13][23]} \sim [m] \qquad [\tilde c_1] = 1$$

$$A_3(+--) = c_2\frac{\langle 23\rangle^3}{\langle 12\rangle\langle 13\rangle} \sim [m] \qquad [c_2] = 1$$

$$A_3(---) = c_3\langle 12\rangle\langle 13\rangle\langle 23\rangle \sim [m]^3 \qquad [c_3] = [m]^{-2}$$

$$A_3(\text{graviton}) = (A_3(\text{gluon}))^2$$

2.35) Consider in gravity an operator constructed by contracting indices on 3 copies of the Riemann tensor; denote this operator R^3.
If we linearize the metric around flat space, $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, we can calculate graviton scattering associated w/ R^3.
What is the mass dimension of the coupling associated w/ R^3?
Use little group scaling to find $A_3(h_1^- h_2^- h_3^-)$ & $A_3(h_1^- h_2^- h_3^+)$.
Compare the result w/ 3-gluon scattering processes.

$$[R] = [\partial]^2 = [m]^2, \qquad [R^3] = [m]^6$$

need $[g_3 R^3] = [m]^4 \Rightarrow [g_3] = [m]^{-2}$

?

$$A_3(---) = g_3(\langle 12\rangle\langle 13\rangle\langle 23\rangle)^2, \qquad A(--+) = \sqrt{g_3}\,\frac{\langle 12\rangle^6}{(\langle 13\rangle\langle 23\rangle)^2}$$

These are the squares of the corresponding gluon amplitudes.

2.36) Consider a dimension-5 Higgs-gluon fusion operator $H\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$, where H is a Higgs scalar field.

Use little group scaling to determine the 3-particle amplitudes of this operator in the limit of massless Higgs, $m_H = 0$.

$$\mathcal{A}_3(g^{h_1} g^{h_2} H) = \begin{cases} c\,\langle 12\rangle^{-h_1-h_2}\langle 13\rangle^{h_2-h_1}\langle 23\rangle^{h_1-h_2} \\ \tilde{c}\,[12]^{h_1+h_2}[13]^{-h_2+h_1}[23]^{-h_1+h_2} \end{cases}$$

$$\mathcal{A}_3(++H) = \tilde{c}_0[12]^2, \quad \mathcal{A}_3(+-H) = \tilde{c}_1\left(\frac{[13]}{[23]}\right)^2,$$

$$\mathcal{A}_3(--H) = c_2\langle 12\rangle^2, \quad \mathcal{A}_3(-+H) = c_1\left(\frac{\langle 13\rangle}{\langle 23\rangle}\right)^2$$

2.37) Why does the 3-pt on-shell amplitude for 3 distinct massless scalars, e.g. $\phi_1\partial_\mu\phi_2\partial^\mu\phi_3$, vanish?

$\phi_1\partial_\mu\phi_2\partial^\mu\phi_3$ gives a vertex $\sim p_2\cdot p_3 = \langle 23\rangle[23]$

but 3-pt amplitudes can't have both angle & square brackets, so it must be 0.

$\phi_1\phi_2\phi_3$ gives a vertex $\sim$ const.

IDK why this one also vanishes...

Lagrangian perspective: $\phi_1\partial_\mu\phi_2\partial^\mu\phi_3 = -\partial^\mu(\phi_1\partial_\mu\phi_2)\phi_3 = -(\partial^\mu\phi_1\partial_\mu\phi_2)\phi_3 - \phi_1(\cancel{\partial^2\phi_2})^{0}\phi_3$

$$= -\phi_2\partial_\mu(\phi_1\partial^\mu\phi_3) = -\phi_2\partial_\mu\phi_1\partial^\mu\phi_3 - \phi_2\phi_1\cancel{\partial^2\phi_3}^{\,0}$$

$$\phi_1\partial_\mu\phi_2\partial^\mu\phi_3 + \phi_2\partial_\mu\phi_1\partial^\mu\phi_3 = 0$$

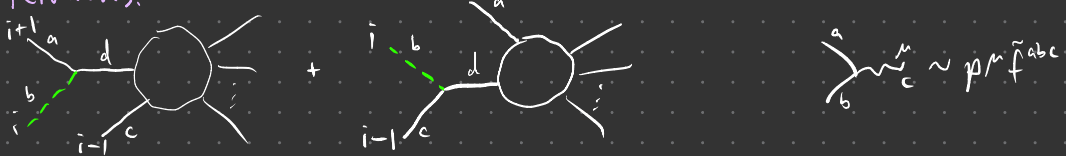
$$\phi_1\partial_\mu\phi_2\partial^\mu\phi_3 + \phi_3\partial_\mu\phi_1\partial^\mu\phi_2 = 0$$

$$\Rightarrow \phi_2\partial_\mu\phi_1\partial^\mu\phi_3 - \phi_3\partial_\mu\phi_1\partial^\mu\phi_2 = 0$$

but we also have $\phi_2\partial_\mu\phi_3\partial^\mu\phi_1 + \phi_3\partial_\mu\phi_1\partial^\mu\phi_2 = 0 \Rightarrow \phi_3\partial_\mu\phi_1\partial^\mu\phi_2 = 0$

etc for other perms of 1, 2, 3

2.38) As a test to see that the soft function is indeed independent of the helicities of the hard legs, compute $A_n[\cdots \phi_{i-1}^{+}\, i^- \,\phi_{i+1}^{-}\cdots]$ where ϕ is a massless adjoint scalar & compare to the version w/ adjoint fermions.



$$\sim \epsilon_{i-}\cdot p_{i+1}\frac{1}{s_{i,i+1}}A_{n-1} - \epsilon_{i-}\cdot p_{i-1}\frac{1}{s_{i-1,i}}A_{n-1} \sim \left(\frac{\langle i|(i+1)|q]}{[qi]\, s_{i,i+1}} - \frac{\langle i|(i-1)|q]}{[qi]\, s_{i-1,i}}\right)A_{n-1}$$

$$= \left(\frac{-\langle i,i+1\rangle[i+1,q]}{[qi]\langle i,i+1\rangle[i,i+1]} + \frac{\langle i,i-1\rangle[i-1,q]}{[qi]\langle i,i-1\rangle[i,i-1]}\right)A_{n-1}$$

$$= \frac{-[i+1,q][i,i-1]+[i-1,q][i,i+1]}{[qi][i,i+1][i,i-1]}A_{n-1} = -\frac{[qi][i-1,i+1]}{[qi][i,i+1][i,i-1]}A_{n-1}$$

$$= \frac{[i+1,i-1]}{[i,i+1][i,i-1]}A_{n-1}$$

off by a sign from fermion version

2.39) Show that the MHV amplitude $A_n[1^+2^+\cdots i^-\cdots j^-\cdots n^+] = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$ has the correct soft behavior.

$$A_n[\cdots i^-\to 0\cdots] = \frac{\langle ij\rangle^4}{\langle 12\rangle\cdots\langle i-1,i\rangle\langle ij\rangle\cdots\langle n1\rangle} = \underbrace{\frac{\langle ij\rangle^3}{\langle i-1,i\rangle}}_{\to 0}\ \underbrace{\frac{1}{\langle 12\rangle\cdots\langle i-2,i-1\rangle\langle j,j+1\rangle\cdots\langle n1\rangle}}_{\text{finite}}$$

$= 0.$ Indeed, $A_{n-1}[1^+2^+\cdots j^-\cdots n^+]=0$

$$A_n[\cdots l^+\to 0\cdots] = \frac{\langle ij\rangle^4}{\langle 12\rangle\cdots\langle l-1,l\rangle\langle l,l+1\rangle\cdots\langle ij\rangle\cdots\langle n1\rangle}$$

$$= \frac{\langle l-1,l+1\rangle}{\langle l-1,l\rangle\langle l,l+1\rangle}\,\frac{\langle ij\rangle^4}{\langle 12\rangle\cdots\langle l-1,l+1\rangle\cdots\langle n1\rangle} = S^+_{l-1,l,l+1}A_{n-1}[1^+2^+\cdots(l-1)^+(l+1)^+\cdots i^-j^-\cdots n^+]$$

3.40) For the MHV amplitude, show that the splitting function for the colinear limit w/ 2 adjacent positive helicity gluons $i, i+1$ is

$$\text{Split}_-(i^+,(i+1)^+) = \frac{1}{\sqrt{z(1-z)}\,\langle i,i+1\rangle}$$

set $|i\rangle = \sqrt{z}\,|P\rangle$, $|i+1\rangle = \sqrt{1-z}\,|P\rangle$

$$A_n[1^-2^-\cdots i^+(i+1)^+\cdots n^+] = \frac{\langle 12\rangle^4}{\langle 12\rangle\cdots\langle i,i+1\rangle\cdots\langle n1\rangle}$$

$-2\cdot\frac{1}{2}-2\frac{1}{2} = -1-1-1-1$

$$= \frac{\langle 12\rangle^4}{\langle 12\rangle\cdots\langle i-1, i\rangle\langle i, i+1\rangle\langle i+1, i+2\rangle\cdots\langle n1\rangle} = \frac{1}{\sqrt{z(1-z)}\,\langle i, i+1\rangle}\,\frac{\langle 12\rangle^4}{\langle 12\rangle\cdots\langle i-1, P\rangle\langle P, i+2\rangle\cdots\langle n1\rangle}$$

$$= \text{Split}_-(i^+, (i+1)^+)\, A_{n-1}[1^- 2^- \cdots P^+ \cdots n^+]$$

$$\text{Split}_-(i^+, (i+1)^+) = \frac{1}{\sqrt{z(1-z)}\,\langle i, i+1\rangle}$$

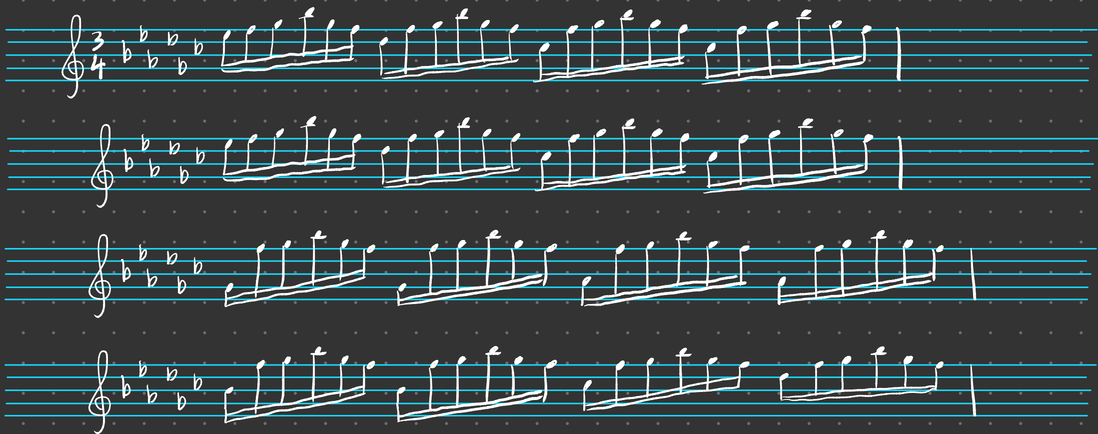
Check that this result is consistent w/ little group scaling.

Check that the result $A_n[1^- 2^- 3^+ \cdots n^+] \xrightarrow{1 \parallel n} \text{Split}_+(n^+, 1^-)\, A_{n-1}[P^- 2^- 3^+ \cdots (n-1)^+]$,

$\text{Split}_+(n^+, 1^-) = \dfrac{z^2}{\sqrt{z(1-z)}\,\langle n1\rangle}$ is consistent w/ little group scaling.

Scaling i & $i+1$ in $i \parallel i+1$ limit: $-2 \cdot \frac{1}{2} - 2 \cdot \frac{1}{2} = -1 - 1$

$t^{-2} A_n = \text{Split}_+(i^+, (i+1)^+; t)\, t^{-1} A_n$ confused...



CHAPTER 3

3.1) Calculate the shift vectors r_i^μ, r_j^μ corresponding to the shift

$$|\hat{i}] = |i] + z|j], \quad |\hat{j}] = |j], \quad |\hat{i}\rangle = |i\rangle, \quad |\hat{j}\rangle = |j\rangle - z|i\rangle$$

$$\hat{p}_i = -|\hat{i}\rangle[\hat{i}| - |\hat{i}]\langle\hat{i}| = -|i\rangle([i| + z[j|) - (|i] + z|j])\langle i|$$

$$= -(|i\rangle[i| + |i]\langle i|) - z(|i\rangle[j| + |j]\langle i|) = p_i + z r_i$$

$$\Rightarrow r_i = -|i\rangle[j| - |j]\langle i| = -|r_i\rangle[r_i| - |r_i]\langle r_i|$$

$$\Rightarrow |r_i] = |j], \quad |r_i\rangle = |i\rangle$$

$$(r_i)^{\dot{a}b} = r_\mu(\bar{\sigma}^\mu)^{\dot{a}b} = -|r_i\rangle^{\dot{a}}[r_i|^b$$

$$r_\mu(\sigma^\nu)_{b\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b} = -2\eta^{\nu\mu}r_\nu = -2r^\mu = -(\sigma^\nu)_{b\dot{a}}|i\rangle^{\dot{a}}[j|^b$$

$$\Rightarrow r_i^\mu = \tfrac{1}{2}(\sigma^\mu)_{b\dot{a}}|i\rangle^{\dot{a}}[j|^b = \tfrac{1}{2}\langle i|\gamma^\mu|j]$$

$$\hat{p}_j = -|\hat{j}\rangle[\hat{j}| - |\hat{j}]\langle\hat{j}| = -(|j\rangle - z|i\rangle)[j| - |j](\langle j| - z\langle i|)$$

$$= -(|j\rangle[j| + |j]\langle j|) + z(|i\rangle[j| + |j]\langle i|) = p_j + z r_j = p_j - z r_i$$

$$\Rightarrow r_j = |i\rangle[j| + |j]\langle i| = -|r_j\rangle[r_j| - |r_j]\langle r_j|$$

$$\Rightarrow |r_j] = -|j], \quad |r_j\rangle = |i\rangle$$

$$\Rightarrow r_j^\mu = \tfrac{1}{2}(\sigma^\mu)_{b\dot{a}}|r_j\rangle^{\dot{a}}[r_j|^b = \tfrac{-1}{2}(\sigma^\mu)_{b\dot{a}}|i\rangle^{\dot{a}}[j|^b = \tfrac{-1}{2}\langle i|\gamma^\mu|j]$$

Check: $r_i^\mu + r_j^\mu = \tfrac{1}{2}\langle i|\gamma^\mu|j] - \tfrac{1}{2}\langle i|\gamma^\mu|j] = 0$

$$r_i \cdot r_j = \tfrac{1}{4}\langle i|\gamma^\mu|j]\langle i|\gamma_\mu|j] = \tfrac{1}{2}\langle ii\rangle[jj] = 0$$

$$p_i \cdot r_i = \tfrac{1}{2}\langle i|i|j] = 0, \quad p_j \cdot r_j = \tfrac{-1}{2}\langle i|j|j] = 0$$

is there a way to write p^μ in terms of $|p\rangle, |p]$

multiple choices of $|p\rangle, |p]$ for any explicit p^μ

(but brackets should be invariant under choice?

3.2) Convince yourself that for large z the Parke-Taylor amplitude falls off as $1/z$ under a $[-,-\rangle$ shift (i.e. choose i & j to be the negative helicity lines). What happens under the other 3 types of shifts?

$$|\hat{1}] = |1] + z|2]$$
$$|\hat{2}\rangle = |2\rangle - z|1\rangle$$

$$\hat{A}_n[1^-2^-3^+\dots n^+] = \frac{\langle\hat{1}\hat{2}\rangle^4}{\langle\hat{1}\hat{2}\rangle\langle\hat{2}3\rangle\cdots\langle n\hat{1}\rangle} = \frac{\langle 12\rangle^4}{\langle 12\rangle(\langle 23\rangle - z\langle 13\rangle)\langle 34\rangle\cdots\langle n1\rangle} \overset{z\to\infty}{\sim} \frac{1}{z}$$

$[1^-, 3^+\rangle$ shift: $|\hat{1}] = |1] + z|3]$, $|\hat{3}\rangle = |3\rangle - z|1\rangle$

$$\hat{A}_n = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 2\hat{3}\rangle\langle \hat{3}4\rangle\cdots\langle n1\rangle} \sim \frac{1}{z^2}$$

$[3^+, 1^-\rangle$ shift: $|\hat{3}] = |3] + z|1]$, $|\hat{1}\rangle = |1\rangle - z|3\rangle$

$$\hat{A}_n = \frac{\langle \hat{1}2\rangle^4}{\langle \hat{1}2\rangle\langle 23\rangle\cdots\langle n\hat{1}\rangle} \sim z^2$$

$[2^-, 3^+\rangle$ shift: $\hat{A}_n \sim \frac{1}{z^2}$

$[3^+, 2^-\rangle$ shift: $\hat{A}_n \sim z^2$

$[3^+, 4^+\rangle$ shift: $\hat{A}_n \sim \frac{1}{z}$

$[4^+, 3^+\rangle$ shift: $\hat{A}_n \sim \frac{1}{z}$

$[3^+, 5^+\rangle$ shift: $\hat{A}_n \sim \frac{1}{z^2}$

$[5^+, 3^+\rangle$ shift: $\hat{A}_n \sim \frac{1}{z^2}$

need to redo these
oops

3.3) Consider a $[1,2\rangle$ shift of the Parke-Taylor amplitude. Identify the simple pole. Calculate the residue of $\hat{A}_n(z)/z$ at this pole.

$$\hat{A}_n(z) = \frac{\langle 12\rangle^4}{\langle 12\rangle(\langle 23\rangle - z\langle 13\rangle)\langle 34\rangle\cdots\langle n1\rangle} = \frac{A_n}{1 - \frac{\langle 13\rangle}{\langle 23\rangle}z} \qquad \text{pole @ } z_0 = \frac{\langle 23\rangle}{\langle 13\rangle}$$

$$\text{Res}\left(\frac{\hat{A}_n(z)}{z}, z = z_0\right) = \text{Res}\left(\frac{A_n}{z(1 - z_0 z)}, z = z_0\right) = \frac{A_n}{z_0} = \frac{\langle 12\rangle^4\langle 13\rangle}{\langle 12\rangle\langle 23\rangle^2\langle 34\rangle\cdots\langle n1\rangle}$$

Verify $A_n = -\sum_{z_I} \text{Res}\left(\frac{\hat{A}_n(z)}{z}, z_I\right) + B_n$

$B_n = \mathcal{O}(z^0)$ term in $z\to\infty$ expansion for $\hat{A}_n(z)$

$\hat{A}_n(z) \sim \frac{1}{z} \to 0$ as $z\to\infty$ $\Rightarrow B_n = 0$

Repeat this for a $[1,3\rangle$ shift.

$$\hat{A}_n(z) = \frac{\langle 12\rangle^4}{\langle 12\rangle(\langle 23\rangle - z\langle 13\rangle)(\langle 34\rangle - z\langle 14\rangle)\langle 45\rangle\cdots\langle n1\rangle} = \frac{A_n}{(1 - \frac{\langle 13\rangle}{\langle 23\rangle}z)(1 - \frac{\langle 14\rangle}{\langle 34\rangle}z)}$$

poles @ $z_0 = \frac{\langle 23\rangle}{\langle 13\rangle}$, $z_1 = \frac{\langle 34\rangle}{\langle 14\rangle}$

$$\text{Res}\left(\frac{\hat{A}_n(z)}{z}, z = z_0\right) = \frac{A_n}{z_0(1 - z_1 z_0)}$$

$$\mathrm{Res}\left(\frac{\hat{A}_n(z)}{z}, z=z_1\right) = \frac{A_n}{z_1(1-z_0z_1)}$$

sum of residues $= \frac{1}{1-z_0z_1}\left(\frac{1}{z_0}+\frac{1}{z_1}\right) = \frac{z_0+z_1}{z_0z_1-(z_0z_1)^2}$

3.4) The spin-1 polarization vectors have denominators $\langle qp\rangle$ & $[qp]$, which may shift under a BCFW shift involving p. Why are there no terms in the on-shell recursion relations corresponding to poles @ $\langle q\hat{p}\rangle = 0$ or $[q\hat{p}]=0$?

If $|\hat{p}\rangle = |p\rangle + z|r\rangle$, $\langle q\hat{p}\rangle = \langle qp\rangle + z\langle qr\rangle = 0$ if $z = -\frac{\langle qp\rangle}{\langle qr\rangle}$

But $|q\rangle$ is not allowed to be $\propto |p\rangle$, so $\langle q\hat{p}\rangle \neq 0$.

(same logic for $[q\hat{p}]$).

3.5) Revisit scalar QED: use little group scaling to determine $A_3(\phi\phi^*\gamma^\pm)$. Then use a $[4,3\rangle$ shift to show that

$$A_4(\phi\phi^*\gamma^+\gamma^-) = g^2\frac{\langle 14\rangle\langle 24\rangle}{\langle 13\rangle\langle 23\rangle}$$

$A_3(\phi\phi^*\gamma^+) = c\langle 12\rangle\langle 13\rangle^{-1}\langle 23\rangle^{-1}$ or $\tilde{c}[12]^{-1}[13][23]$

Need $[A_3] = [m] \Rightarrow [\tilde{c}] = [m]^0$ Indeed, $[\tilde{e}] = [m]^0$ in scalar QED

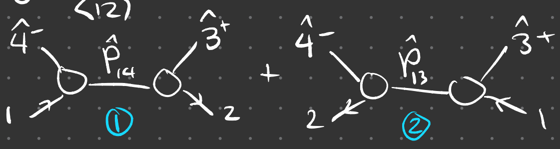
$\Rightarrow A_3(\phi\phi^*\gamma^+) = \tilde{e}\frac{[13][23]}{[12]}$

$A_3(\phi\phi^*\gamma^-) = c\langle 12\rangle^{-1}\langle 13\rangle\langle 23\rangle$ or $\hat{c}[12][13]^{-1}[23]^{-1}$

again need $[c] = [m]^0$ $|\hat{4}] = |4] + z|3]$

$\Rightarrow A_3(\phi\phi^*\gamma^-) = \tilde{e}\frac{\langle 13\rangle\langle 23\rangle}{\langle 12\rangle}$ $I = \{4,1\}, \{4,2\}$

$A_4(\phi\phi^*\gamma^+\gamma^-) =$



internal line can be ϕ, ϕ^*, or $\gamma^\pm$

if ϕ: ① $= \hat{A}_3(\underset{1}{-\hat{P}_{14}^{(\phi^*)}}\,\underset{\hat{4}}{\phi\gamma^-})\frac{1}{P_{14}^2}\hat{A}_3(\underset{2}{\phi^*}\,\underset{\hat{3}}{\hat{P}_{14}^{(\phi)}\gamma^+})$

$\rightarrow$

$$= \tilde{e}^2 \frac{\langle \hat{P}_{14}\hat{4}\rangle\langle 1\hat{4}\rangle}{\langle \hat{P}_{14}1\rangle} \frac{1}{\langle 14\rangle[14]} \frac{[2\hat{3}][\hat{P}_{14}\hat{3}]}{[2\hat{P}_{14}]}$$

$$\langle \hat{P}\hat{4}\rangle[\hat{P}\hat{3}] = \langle \hat{4}|\hat{P}_{14}|\hat{3}] = \langle \hat{4}|1|\hat{3}]$$
$$= \langle 41\rangle[31]$$

$$\langle \hat{P}1\rangle[2\hat{P}] = -\langle 1|\hat{P}|2] = -\langle 1|(\hat{p}_1+\hat{p}_4)|2] = -\langle 1|\hat{p}_4|2]$$
$$= \langle 1\hat{4}\rangle[\hat{4}2] = \langle 14\rangle[\hat{4}2]$$

$$[\hat{4}2] = [42] + z[32]$$

$$① = \tilde{e}^2 \frac{\langle 14\rangle[23]\langle 14\rangle[13]}{\langle 14\rangle[14]\langle 14\rangle([42]+z[32])} = -\tilde{e}^2 \frac{[13][23]}{[14]([24]+z[23])}$$

evaluate @ $z = z_{14}$ $\qquad 0 = \hat{P}_{14}^2 = \langle 14\rangle[1\hat{4}] \Rightarrow [1\hat{4}] = 0 = [14] + z_{14}[13]$

$$[24] + z_{14}[23] = [24] - \frac{[14]}{[13]}[23] = \frac{[24][13]-[14][23]}{[13]} \qquad \Rightarrow z_{14} = -\frac{[14]}{[13]}$$

$$= \frac{[12][34]}{[13]} \qquad [24][13]+[21][34]+[23][41]=0$$

$$①(z=z_{14}) = -\tilde{e}^2 \frac{[13]^2[23]}{[14][12][34]}$$

$$② = \hat{A}_3(\underset{2}{-\hat{P}_{13}^{(\phi)}}\phi^* \underset{4}{\gamma^-}) \frac{1}{P_{13}^2} \hat{A}_3(\underset{1}{\phi}P_{13}^{(\phi^*)}\underset{3}{\gamma^+})$$

$$= \tilde{e}^2 \frac{\langle \hat{P}_{13}\hat{4}\rangle\langle 2\hat{4}\rangle}{\langle \hat{P}_{13}2\rangle} \frac{1}{\langle 13\rangle[13]} \frac{[1\hat{3}][\hat{P}_{13}\hat{3}]}{[1\hat{P}_{13}]} \qquad |\hat{3}\rangle = |3\rangle - z|4\rangle$$

$$\langle \hat{P}_{13}\hat{4}\rangle[\hat{P}_{13}\hat{3}] = \langle 4|\hat{P}_{13}|3] = \langle 4|1|3] = \langle 41\rangle[31]$$

$$\langle \hat{P}_{13}2\rangle[1\hat{P}_{13}] = -\langle 2|\hat{P}_{13}|1] = -\langle 2|\hat{3}|1] = \langle 2\hat{3}\rangle[31]$$
$$= [31](\langle 23\rangle - z\langle 24\rangle)$$

$$② = \tilde{e}^2 \frac{\langle 24\rangle[13]\langle 14\rangle[13]}{\langle 13\rangle[13](\langle 23\rangle - z\langle 24\rangle)[31]} = -\tilde{e}^2 \frac{\langle 14\rangle\langle 24\rangle}{\langle 13\rangle(\langle 23\rangle - z\langle 24\rangle)}$$

evaluate @ $z = z_{13}$ $\qquad 0 = \hat{P}_{13}^2 = \langle 1\hat{3}\rangle[13] \Rightarrow \langle 1\hat{3}\rangle = 0 = \langle 13\rangle - z_{13}\langle 14\rangle$

$$\Rightarrow z_{13} = \frac{\langle 13\rangle}{\langle 14\rangle}$$

$$\langle 23\rangle - \frac{\langle 13\rangle}{\langle 14\rangle}\langle 24\rangle = \frac{\langle 23\rangle\langle 14\rangle - \langle 13\rangle\langle 24\rangle}{\langle 14\rangle} = \frac{-\langle 12\rangle\langle 34\rangle}{\langle 14\rangle}$$

$$②(z=z_{13}) = \tilde{e}^2 \frac{\langle 14\rangle^2\langle 24\rangle}{\langle 13\rangle\langle 12\rangle\langle 34\rangle}$$

P can't be γ or ϕ^* b/c $A_3(\phi\phi\gamma)=0$, $A_3(\phi\gamma\gamma)=0$

$$\Rightarrow A_4 = ① + ② = \tilde{e}^2\left(\frac{-[13]^2[23]}{[12][14][34]} + \frac{\langle 14\rangle^2\langle 24\rangle}{\langle 12\rangle\langle 13\rangle\langle 34\rangle}\right)$$

has correct little gp scaling ✓

$$= \tilde{e}^2\left(\frac{-\langle 12\rangle\langle 13\rangle\langle 34\rangle[13]^2[23] + \langle 14\rangle^2\langle 24\rangle[12][14][34]}{\langle 12\rangle[12]\langle 34\rangle[34]\langle 13\rangle[14]}\right)$$

$$= \tilde{e}^2\,\frac{\cancel{\langle 12\rangle}[23]\langle 24\rangle\cancel{[12]}\left(\langle 13\rangle[13]+\langle 14\rangle[14]\right)}{\cancel{\langle 12\rangle}\cancel{[12]}\langle 34\rangle[34]\langle 13\rangle[14]}$$

$\langle 34\rangle[13] = -\langle 24\rangle[12]$

$\langle 12\rangle[23] = \langle 14\rangle[34]$

$$= -\tilde{e}^2\,\frac{\cancel{\langle 12\rangle[12]}\langle 24\rangle[23]}{\cancel{\langle 12\rangle[12]}\langle 13\rangle[14]} = -\tilde{e}^2\frac{\langle 14\rangle\langle 24\rangle}{\langle 13\rangle\langle 23\rangle}$$

$\frac{[23]}{[14]} = \frac{\langle 14\rangle}{\langle 23\rangle}$

off by a (-)... oh well (check analytic cont.)

What is the large-z fall off of this amplitude under a $[4,3\rangle$ shift?

Before plugging in values for z, both diagrams had the form $\frac{(\ \ )}{(1\pm(\ \ )z)} \sim \pm\frac{1}{z}$ as $z\to\infty$, so the shift is valid.

3.6) Calculate the 4-graviton amplitude $\mathcal{M}_4(1^-2^-3^+4^+)$ by first fixing $\mathcal{M}_3$ by little gp scaling and then using a $[1,2\rangle$ shift. Check the little gp scaling & Bose symmetry of $\mathcal{M}_4$.

$$A_3(+++) = g^5\left([12][23][31]\right)^2$$

where $[g]=[m]^{-1}$

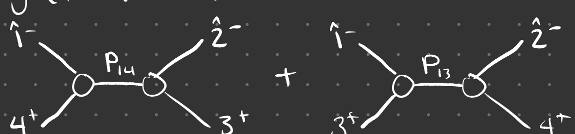
$$A_3(++-) = g\,\frac{[12]^6}{(\langle 13\rangle\langle 23\rangle)^2}$$

$$A_3(+--) = g\,\frac{\langle 23\rangle^6}{\langle 12\rangle^2\langle 13\rangle^2}$$

$$A_3(---) = g^5\left(\langle 12\rangle\langle 13\rangle\langle 23\rangle\right)^2$$

from (2.34)

Diagrams:



$|\hat{1}] = |1] + z|2]$

$|\hat{2}\rangle = |2\rangle - z|1\rangle$

① ② = ① w/ $3\leftrightarrow 4$

$$①:\ \hat{A}_3(4^+\hat{1}^-{-\hat{P}_{14}^-})\frac{1}{P_{14}^2}\hat{A}_3(3^+\hat{P}_{14}^+\hat{2}^-) + \hat{A}_3(-\hat{P}_{14}^+4^+1^-)\frac{1}{P_{14}^2}\hat{A}_3(3^+\hat{P}_{14}^-\hat{2}^-)$$

$$= g^2\frac{1}{\langle 14\rangle[14]}\left(\frac{\langle\hat{1}\hat{P}_{14}\rangle^6}{\langle 4\hat{1}\rangle^2\langle 4\hat{P}_{14}\rangle^2}\,\frac{[3\hat{P}_{14}]^6}{[3\hat{2}]^2[\hat{P}_{14}\hat{2}]^2} + \cancel{\frac{[\hat{1}\hat{P}_{14}]^6}{[\hat{P}_{14}\hat{1}]^2[4\hat{1}]^2}\,\frac{\langle\hat{P}_{14}\hat{2}\rangle^6}{\langle 3\hat{P}_{14}\rangle^2\langle 3\hat{2}\rangle^2}}^{\,0}\right)$$

$\langle\hat{2}\hat{P}_{14}\rangle[\hat{1}\hat{P}_{14}]=0$

$[\hat{1}4]=0$

$\langle\hat{2}3\rangle=0$

$$= g^2 \frac{\langle 14\rangle^6[34]^6}{\langle 14\rangle^3[14][12]^2\langle 14\rangle^2[23]^2} = g^2 \frac{\langle 14\rangle[34]^6}{[14][12]^2[23]^2}$$

$\langle \hat{1}\hat{P}_{14}\rangle[3\hat{P}_{14}] = \langle 14\rangle[34]$

$\langle 4\hat{P}_{14}\rangle[\hat{P}_{14}\hat{2}] = -\langle 14\rangle[12]$

$$\mathcal{M}_4 = g^2\left(\frac{\langle 14\rangle[34]^6}{[14][12]^2[23]^2} + \frac{\langle 13\rangle[34]^6}{[13][12]^2[24]^2}\right)$$

symmetric under $1 \leftrightarrow 2$, $3 \leftrightarrow 4$

$h_1 = h_2 = -2$, $h_3 = h_4 = 2$

$\left.\begin{array}{ll} 4 = 1+1+2 & \text{for } 1^-, 2^- \\ -4 = -6+2 & \text{for } 3^+, 4^+ \end{array}\right\} \to$ correct little gp scaling

Show that $\mathcal{M}_4$ satisfies the 4-pt KLT relations

$\mathcal{M}_4(1234) = -s_{12}A_4[1234]A_4[1243]$ where A_4 is the Park-Taylor amplitude & s_{12} is the Mandelstam variable $s_{12} = -(p_1+p_2)^2$.

Want to show $\mathcal{M}_4(1^-2^-3^+4^+) = -g^2\langle 12\rangle[12]\dfrac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\dfrac{\langle 12\rangle^4}{\langle 12\rangle\langle 24\rangle\langle 43\rangle\langle 31\rangle}$

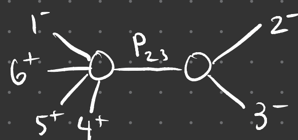
$$\mathcal{M}_4 = g^2\frac{[34]^6(\langle 14\rangle[13][24]^2 + \langle 13\rangle[14][23]^2)}{[14][13][12]^2[23]^2[24]^2} = g^2\frac{[34]^6\langle 14\rangle[24][12][34]}{[14][13][12]^2[23]^2[24]^2}$$

$$= g^2\frac{[34]^6\langle 14\rangle\langle 12\rangle}{[14][13][23]^2[24]\langle 34\rangle} = g^2\frac{\langle 12\rangle^8\ \langle 12\rangle[12]}{\langle 14\rangle\langle 23\rangle\langle 24\rangle\langle 13\rangle\langle 34\rangle\langle 12\rangle^2\langle 34\rangle}$$

$$= -g^2\langle 12\rangle[12]\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 24\rangle\langle 43\rangle\langle 31\rangle}$$

$\left(\text{using } [34] = \dfrac{\langle 12\rangle[23]}{\langle 14\rangle} = \dfrac{-\langle 12\rangle[13]}{\langle 24\rangle} = \dfrac{-\langle 12\rangle[24]}{\langle 13\rangle} = \dfrac{\langle 12\rangle[14]}{\langle 23\rangle} = \dfrac{\langle 12\rangle[12]}{\langle 34\rangle}\right)$

3.7) Consider the split-helicity NMHV amplitude $A_6[1^-2^-3^-4^+5^+6^+]$. Show that, under a $[1,2\rangle$ shift, the 23-channel diagram does not contribute.



$= A_5[\hat{1}^- -\hat{P}^- 4^+5^+6^+]\dfrac{1}{P^2}A_3[\hat{P}^+ 2^- 3^-]$

(other helicity for P has $A_5 = 0$)

$$= \frac{\langle \hat{1}\hat{P}\rangle^4}{\langle \hat{1}\hat{P}\rangle\langle \hat{P}4\rangle\langle 45\rangle\langle 56\rangle\langle 6\hat{1}\rangle}\ \frac{1}{\langle 23\rangle[23]}\ \frac{\langle \hat{2}3\rangle^3}{\langle \hat{P}\hat{2}\rangle\langle \hat{P}3\rangle}$$

$P^2 = \langle \hat{2}3\rangle[23] = 0 \Rightarrow \langle \hat{2}3\rangle = 0$

$\left.\begin{array}{l} |\hat{P}]\langle \hat{P}3\rangle = -\hat{P}|3\rangle = -\hat{p}_2|3\rangle = |\hat{2}]\langle \hat{2}3\rangle = 0 \\ \Rightarrow \langle \hat{P}3\rangle = 0 \\ |\hat{P}]\langle \hat{P}\hat{2}\rangle = -\hat{P}|\hat{2}\rangle = -p_3|\hat{2}\rangle = |3]\langle 3\hat{2}\rangle = 0 \\ \Rightarrow \langle \hat{P}\hat{2}\rangle = 0 \end{array}\right\} \to A_3[\hat{P}^+2^-3^-] = 0$

$|\hat P]\langle \hat 1 \hat P\rangle = \hat P|1\rangle = \hat p_2|1\rangle + p_3|1\rangle = |2]\langle 21\rangle + |3]\langle 31\rangle \neq 0$

$|\hat P]\langle \hat P 4\rangle = -\hat P|4\rangle = -\hat p_2|4\rangle - p_3|4\rangle = -|2]\langle \hat 2 4\rangle - |3]\langle 34\rangle \neq 0$

$\Rightarrow A_5[\hat 1^- \hat P^- 4^+ 5^+ 6^+]$ is finite

3.8) Check the little gp scaling of the 6-pt split-helicity result

$$A_6[---+++] = \frac{\langle 3|1+2|6]^3}{P_{126}^2[21][16]\langle 34\rangle\langle 45\rangle\langle 5|1+6|2]} + \frac{\langle 1|5+6|4]^3}{P_{156}^2[23][34]\langle 56\rangle\langle 61\rangle\langle 5|1+6|2]}$$

1st term: $2 = 1+1$ ✓ $\quad -2 = -1-1$ ✓

$2 = 1+1$ ✓ $\quad -2 = -1-1$ ✓

$2 = 3-1$ ✓ $\quad -2 = -3+1$ ✓

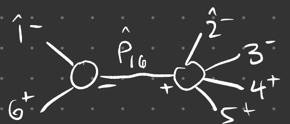
2nd term: $2 = 3-1$ ✓ $\quad -2 = -3+1$ ✓

$2 = 1+1$ ✓ $\quad -2 = -1-1$ ✓

$2 = 1+1$ ✓ $\quad -2 = -1-1$ ✓

Fill in the details to derive this from the $[1,2\rangle$ shift.

Diagram ①:



$= A_3[6^+ \hat 1^- -\hat P^-]\frac{i}{P^2} A_5[\hat 2^- 3^- 4^+ 5^+ \hat P^+]$

$$= \frac{\langle \hat 1 \hat P\rangle^3}{\langle 6\hat 1\rangle\langle 6\hat P\rangle}\,\frac{1}{\langle 16\rangle[16]}\,\frac{\langle \hat 2 3\rangle^4}{\langle \hat 2 3\rangle\langle 34\rangle\langle 45\rangle\langle 5\hat P\rangle\langle \hat P\hat 2\rangle}$$

$\langle 16\rangle[\hat 1 6] = 0 \Rightarrow [\hat 1 6] = 0 = [16] + z_{16}[26]$

$[\hat 1 \hat P] = [\hat P 6] = 0$

$|\hat P]\langle \hat 1 \hat P\rangle = p_6|1\rangle = |6]\langle 16\rangle$

$|\hat P]\langle 6\hat P\rangle = \hat p_1|6\rangle = |\hat 1]\langle 61\rangle$

$|\hat P]\langle 5\hat P\rangle = \hat p_1|5\rangle + p_6|5\rangle = |\hat 1]\langle 51\rangle + |6]\langle 56\rangle$

$|\hat P]\langle \hat P\hat 2\rangle = -\hat p_1|\hat 2\rangle - p_6|\hat 2\rangle = |\hat 1]\langle 12\rangle + |6]\langle 6\hat 2\rangle$

multiply by $[2\hat P]^3/[2\hat P]^3$:

$$① = \frac{[26]^3\langle \hat 2 3\rangle^3}{[21][16]\langle 34\rangle\langle 45\rangle\langle 5|1+6|2]([21]\langle 12\rangle + [26]\langle 6\hat 2\rangle)}$$

evaluate @ $z = z_{16} = \frac{-[16]}{[26]}$:

$$\langle \hat 2 3\rangle = \langle 23\rangle - z_{16}\langle 13\rangle = \frac{\langle 23\rangle[26] + [16]\langle 13\rangle}{[26]} = \frac{\langle 3|2+1|6]}{[26]}$$

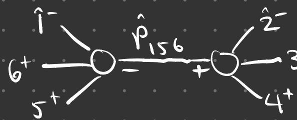
$$\langle 6\hat 2\rangle = \langle 62\rangle - z_{16}\langle 61\rangle = \frac{\langle 62\rangle[26] + [16]\langle 61\rangle}{[26]}$$

$$[21]\langle 12\rangle + [26]\langle 6\hat{2}\rangle = -\langle 12\rangle[12] - \langle 26\rangle[26] - \langle 16\rangle[16]$$

$$= -\left((p_1+p_2)^2+(p_2+p_6)^2+(p_1+p_6)^2\right) = -(p_1+p_2+p_6)^2 = -P_{126}^2$$

$$① = \frac{\langle 3|1+2|6]^3}{P_{126}^2 [21][61]\langle 34\rangle\langle 45\rangle\langle 5|1+6|2]}$$

Diagram ②:



$$= A_4[\hat{1}^- -\hat{p}^- 5^+ 6^+]\frac{1}{p^2} A_4[\hat{2}^- 3^- 4^+ \hat{p}^+]$$

$$= \frac{\langle 1\hat{p}\rangle^3}{\langle \hat{p}5\rangle\langle 56\rangle\langle 61\rangle}\frac{1}{p^2}\frac{\langle \hat{2}3\rangle^3}{\langle 34\rangle\langle 4\hat{p}\rangle\langle \hat{p}\hat{2}\rangle}$$

$$\hat{P}_{156}^2 = \langle 15\rangle[\hat{1}5] + \langle 16\rangle[\hat{1}6] + \langle 56\rangle[56] = 0$$

$$= \langle 15\rangle([15] + z_{156}[25]) + \langle 16\rangle([16] + z_{156}[26]) + \langle 56\rangle[56]$$

$$z_{156}\langle 1|5+6|2] = -p^2 \qquad \Rightarrow \qquad z_{156} = -\frac{p^2}{\langle 1|5+6|2]}$$

$$|\hat{p}]\langle 1\hat{p}\rangle = (p_5+p_6)|1\rangle$$

$$|\hat{p}]\langle \hat{p}5\rangle = -(\hat{p}_1+p_6)|5\rangle$$

$$|\hat{p}]\langle 4\hat{p}\rangle = (\hat{p}_1+p_5+p_6)|4\rangle$$

$$|\hat{p}]\langle \hat{p}\hat{2}\rangle = (p_3+p_4)|\hat{2}\rangle$$

$$[4\hat{p}]\langle 4\hat{p}\rangle = (\hat{p}+p_4)^2 = (\hat{p}_2+p_3)^2 = \langle \hat{2}3\rangle[23]$$

$$[4\hat{p}]\langle \hat{p}\hat{2}\rangle = [4|3|\hat{2}\rangle = \langle \hat{2}3\rangle[43]$$

multiply by $[4\hat{p}]^3/[4\hat{p}]^3$:

$$② = \frac{\langle 1|5+6|4]^3\langle \hat{2}3\rangle}{p^2\langle 5|\hat{1}+6|4]\langle 56\rangle\langle 61\rangle\langle 34\rangle[23][34]}$$

evaluate @ $z = z_{156}$:

$$\langle \hat{2}3\rangle = \langle 23\rangle - z_{156}\langle 13\rangle = \langle 23\rangle + \frac{p^2}{\langle 1|5+6|2]}\langle 13\rangle$$

$$= \frac{\langle 23\rangle\langle 15\rangle[25] + \langle 23\rangle\langle 16\rangle[26] + \langle 13\rangle\langle 15\rangle[15] + \langle 13\rangle\langle 16\rangle[16] + \langle 13\rangle\langle 56\rangle[56]}{\langle 1|5+6|2]}$$

$$\langle 5|\hat{1}+6|4] = \langle 5|1+6|4] + \langle 51\rangle[24]\frac{p^2}{\langle 1|5+6|2]}$$

$$= \frac{\langle 5|1+6|4]\langle 1|5+6|2] - \langle 15\rangle[24]\left(\langle 15\rangle[15] + \langle 16\rangle[16] + \langle 56\rangle[56]\right)}{\langle 1|5+6|2]}$$

[redacted] $= (\langle 15\rangle[14]+\langle 56\rangle[46])(\langle 15\rangle[25]+\langle 16\rangle[26])$

$= \langle15\rangle\langle15\rangle[14][25]+\langle15\rangle\langle16\rangle[14][26]+\langle56\rangle\langle15\rangle[46][25]+\langle56\rangle\langle16\rangle[46][26]$

$$\langle 5|\hat 1+6|4]\langle 1|5+6|2] = \langle15\rangle\langle15\rangle[12][45]+\langle15\rangle\langle16\rangle[12][46] + \langle15\rangle\langle56\rangle[45][26]+\langle16\rangle\langle56\rangle[26][46]$$

$$=(\langle15\rangle[12]+\langle56\rangle[26])(\langle15\rangle[45]+\langle16\rangle[46])$$

$$=\langle5|1+6|2]\langle1|5+6|4]$$

$$\frac{\langle\hat23\rangle}{\langle34\rangle} \stackrel{?}{=} \frac{\langle1|5+6|4]}{\langle1|5+6|2]}$$

$$\langle56\rangle[56]=(p_5+p_6)^2 = (p_1+p_2+p_3+p_4)^2$$

$$\langle34\rangle\langle1|5+6|4] = \langle34\rangle\langle15\rangle[45]+\langle34\rangle\langle16\rangle[46]$$

$$\langle\hat23\rangle\langle1|5+6|2] = \langle23\rangle\langle15\rangle[25]+\langle23\rangle\langle16\rangle[26]+\langle13\rangle\langle15\rangle[15]+\langle13\rangle\langle16\rangle[16]+\langle13\rangle\langle56\rangle[56]$$

$$=\langle15\rangle\langle3|1+2|5]+\langle16\rangle\langle3|1+2|6]+\langle13\rangle\langle56\rangle[56]$$

$$=-\langle15\rangle\langle3|4+6|5]-\langle16\rangle\langle3|4+5|6]+\langle13\rangle\langle56\rangle[56]$$

$$=\langle15\rangle\langle34\rangle[45]-\langle15\rangle\langle36\rangle[56]+\langle16\rangle\langle34\rangle[46]+\langle16\rangle\langle35\rangle[56]+\langle13\rangle\langle56\rangle[56]$$

$$=\langle15\rangle\langle34\rangle[45]+\langle16\rangle\langle34\rangle[46]$$

$$\Rightarrow \frac{\langle\hat23\rangle}{\langle34\rangle}=\frac{\langle1|5+6|4]}{\langle1|5+6|2]}$$

Finally, (2) $= \dfrac{\langle1|5+6|4]^3}{P_{156}^2[23][34]\langle56\rangle\langle61\rangle\langle5|1+6|2]}$

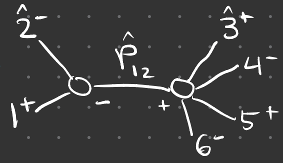
3.9) Show that the BCFW recursion relations based on the $[2,3\rangle$ shift give the following representation of the 6-pt alternating-helicity amplitude $A_6[1^+2^-3^+4^-5^+6^-] = M_2+M_4+M_6$ where

$$M_i = \frac{\langle i,i+2\rangle^4[i+3,i-1]^4}{\tilde P_i^2\langle i|\tilde P_i|i+3]\langle i+2|\tilde P_i|i-1]\langle i,i+1\rangle\langle i+1,i+2\rangle[i+3,i-2][i-2,i-1]}$$

and $\tilde P_i = P_{i,i+1,i+2}$.

$$M_4 = \frac{\langle46\rangle^4[13]^4}{P_{456}^2\langle4|P|1]\langle6|P|3]\langle45\rangle\langle56\rangle[12][23]}$$

① :



$\langle 12\rangle[1\hat{2}] = 0 \Rightarrow [1\hat{2}] = 0$

$\Rightarrow [1\hat{P}] = [\hat{2}\hat{P}] = 0$

$$= A_3[1^+\hat{2}^- -\hat{P}^-]\,\frac{1}{P^2}\,A_5[\hat{P}^+\hat{3}^+4^-5^+6^-]$$

$$= \frac{1}{P^2}\,\frac{\langle 2\hat{P}\rangle^3}{\langle 12\rangle\langle 1\hat{P}\rangle}\,\frac{\langle 46\rangle^4}{\langle\hat{P}\hat{3}\rangle\langle\hat{3}4\rangle\langle 45\rangle\langle 56\rangle\langle 6\hat{P}\rangle}$$

the last person to try this

$|\hat{P}]\langle 2\hat{P}\rangle = |1]\langle 21\rangle$

$|\hat{P}]\langle 1\hat{P}\rangle = |\hat{2}]\langle 12\rangle$

$|\hat{P}]\langle\hat{P}\hat{3}\rangle = -\hat{P}_{12}|\hat{3}\rangle = \hat{P}_{3456}|\hat{3}\rangle = \hat{P}_{456}|\hat{3}\rangle = |4]\langle\hat{3}4\rangle + |5]\langle\hat{3}5\rangle + |6]\langle\hat{3}6\rangle$

$|\hat{P}]\langle 6\hat{P}\rangle = |1]\langle 61\rangle + |\hat{2}]\langle 62\rangle$

3.10) Show that the square-spinor shift, $|\hat{i}] = |i] + z c_i |X]$, $|\hat{i}\rangle = |i\rangle$ w/ c_i s.t. $\sum_{i=1}^{n} c_i |i\rangle = 0$, gives shift vectors r_i^μ that satisfy i) $\sum_i r_i^\mu = 0$, ii) $r_i \cdot r_j = 0 \;\; \forall\, i,j \in \{1,\dots,n\}$, iii) $p_i \cdot r_i = 0 \;\; \forall\, i \in \{1 \dots n\}$.

$$\hat{\not{p}}_i = -|\hat{i}\rangle[\hat{i}| - |\hat{i}]\langle\hat{i}| = -|i\rangle([i| + z c_i [X|) - (|i] + z c_i |X])\langle i|$$

$$= -|i\rangle[i| - |i]\langle i| - z c_i (|i\rangle[X| + |X]\langle i|) = \not{p}_i + z \not{r}_i$$

$$\Rightarrow \not{r}_i = -c_i(|i\rangle[X| - |X]\langle i|) = -|r_i\rangle[r_i| - |r_i]\langle r_i|$$

$$\Rightarrow |r_i\rangle = |i\rangle, \quad |r_i] = c_i |X]$$

$$\Rightarrow r_i^\mu = \tfrac{1}{2}\langle r_i|\gamma^\mu|r_i] = \tfrac{1}{2} c_i \langle i|\gamma^\mu|X]$$

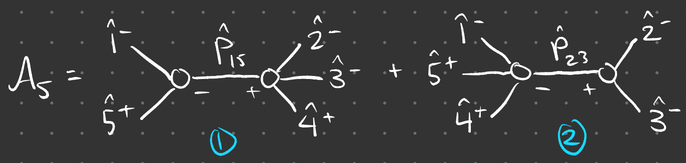
$$\sum r_i^\mu = \tfrac{1}{2}\sum c_i \langle i|\gamma^\mu|X] = \tfrac{1}{2}\sum [X|\gamma^\mu \sum c_i |i\rangle = 0$$

$$r_i \cdot r_j = \tfrac{1}{2} c_i \langle i|\gamma^\mu|X] \tfrac{1}{2} c_j \langle j|\gamma_\mu|X] \sim \langle ij\rangle[XX] = 0$$

$$p_i \cdot r_i = \tfrac{1}{2} c_i \langle i|i|X] = 0$$

3.11) Construct $A_5[1^-2^-3^-4^+5^+]$ from the CSW expansion. Make a choice for the reference spinor $|X]$ to simplify the calculation & show that the result agrees w/ the anti-MHV Parke-Taylor formula.

$$A_5 = \frac{[45]^3}{[12][23][34][51]}$$



① $= A_3[\hat{5}^+ \hat{1}^- \, -\hat{P}_{15}^-] \frac{1}{P_{15}^2} A_4[\hat{2}^-\hat{3}^-\hat{4}^+\hat{P}_{15}^+] = \frac{\langle 1\hat{P}\rangle^3}{\langle 5\hat{P}\rangle\langle 51\rangle} \frac{1}{\langle 15\rangle[15]} \frac{\langle 23\rangle^3}{\langle 34\rangle\langle 4\hat{P}\rangle\langle\hat{P}2\rangle}$

$$= \frac{\langle 1|5|X]^3 \langle 23\rangle^3}{\langle 5|1|X]\langle 4|1+5|X]\langle 2|1+5|X]\langle 15\rangle^2\langle 34\rangle[15]}$$

② $= A_4[\hat{1}^- \, -\hat{P}_{23}^- \, \hat{4}^+ \hat{5}^+] \frac{1}{P_{23}^2} A_3[\hat{P}_{23}^+ \hat{2}^- \hat{3}^-] = \frac{\langle 1\hat{P}\rangle^3}{\langle\hat{P}4\rangle\langle 45\rangle\langle 51\rangle} \frac{1}{\langle 23\rangle[23]} \frac{\langle 23\rangle^3}{\langle\hat{P}2\rangle\langle\hat{P}3\rangle}$

$$= \frac{\langle 1|2+3|5]^3 \langle 23\rangle^3}{\langle 4|2+3|X]\langle 2|3|X]\langle 3|2|X]\langle 23\rangle\langle 45\rangle\langle 15\rangle[23]}$$

choose $|X] = |5] \Rightarrow (1) = 0$

LG scaling:

$2 = 2+1-1 \qquad -2 = 2-3-1$

$2 = 1+1 \qquad -2 = -3+3-2$ ✓

$2 = 1+1$

$$A_5 = \frac{\langle 1|4|5]^3 \langle 23\rangle^2}{\langle 4|1|5]\langle 2|3|5]\langle 3|2|5]\langle 45\rangle\langle 15\rangle[23]}$$

$$= \frac{\langle 14\rangle^2 [45]^3}{[15][35][25]\langle 45\rangle\langle 15\rangle[23]}$$

$[12][23][34][51]$

$$\frac{\langle 14\rangle^2}{[35][25]\langle 45\rangle\langle 15\rangle} \stackrel{?}{=} \frac{1}{[12][34]}$$

$$\langle 14\rangle^2[12][34] = \langle 4|1|2]\langle 1|4|3] = \langle 4|3+5|2]\langle 1|2+5|3]$$

$$= (\langle 43\rangle[23] + \langle 4|5|2])(\langle 12\rangle[32] + \langle 1|5|3])$$

$$= \underbrace{\langle 34\rangle\langle 12\rangle[23]^2 + \langle 43\rangle[23]\langle 15\rangle[35] + \langle 45\rangle[25]\langle 12\rangle[32]}_{(-\langle 4|3|2]\langle 12\rangle \; -\langle 4|3|5]\langle 15\rangle \; -\langle 45\rangle[25]\langle 12\rangle)[23]} + \langle 1|5|3]\langle 4|5|2]$$

$$\sim \langle 4|3(|2]\langle 2| + |5]\langle 5|)|1\rangle - \langle 45\rangle[25]\langle 12\rangle$$

$$= -\langle 4|3|4]\langle 41\rangle - \langle 45\rangle[25]\langle 12\rangle$$

$$= \langle 34\rangle[34]\langle 14\rangle - \langle 45\rangle[25]\langle 12\rangle$$

$$= \langle 14\rangle\langle 12\rangle[12] + \langle 14\rangle\langle 15\rangle[15] + \langle 14\rangle\langle 25\rangle[25] - \langle 12\rangle\langle 45\rangle[25]$$

$$= \langle 15\rangle\langle 24\rangle[25] + \langle 14\rangle\langle 12\rangle[12] + \langle 14\rangle\langle 15\rangle[15]$$

$$\langle 12\rangle(-\langle 4|3|2] - \langle 4|5|2]) - \langle 4|3|5]\langle 15\rangle$$

$$= \langle 12\rangle\langle 14\rangle[12] - \langle 34\rangle[35]\langle 15\rangle$$

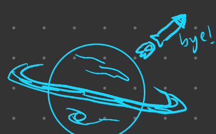
$$= \langle 14\rangle\langle 34\rangle[34] + \langle 14\rangle\langle 35\rangle[35] + \langle 14\rangle\langle 45\rangle[45] - \langle 15\rangle\langle 34\rangle[35]$$

$$\langle 14\rangle\langle 35\rangle - \langle 15\rangle\langle 34\rangle = \langle 13\rangle\langle 45\rangle$$

$$= \langle 14\rangle\langle 34\rangle[34] - \langle 1|3+4|5]\langle 45\rangle = \langle 14\rangle\langle 34\rangle[34] + \langle 1|2|5]\langle 45\rangle$$

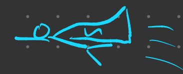
$$= \langle 14\rangle\langle 34\rangle[34] - \langle 12\rangle\langle 45\rangle[25]$$

... stuck :(



CHAPTER 4

$\psi^\dagger_{\dot a}\,\bar\sigma^{\mu \dot a b}\,\psi_b$



4.1) Consider the chiral model $\mathcal{L}_0 = i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi - \partial_\mu\bar\phi\,\partial^\mu\phi$ for a Weyl fermion ψ & complex scalar ϕ. Show that $\mathcal{L}$ is invariant (up to a total derivative) under the transformation

$$\begin{cases} \delta_\varepsilon\phi = \varepsilon\psi, & \delta_\varepsilon\bar\phi = \varepsilon^\dagger\psi^\dagger \\ \delta_\varepsilon\psi_a = -i(\sigma^\mu)_{a\dot b}(\varepsilon^\dagger)^{\dot b}\partial_\mu\phi, & \delta_\varepsilon(\psi^\dagger)_{\dot a} = i(\partial_\mu\bar\phi)\varepsilon^b(\sigma^\mu)_{b\dot a} \end{cases}$$

$$\delta\mathcal{L}_0 = i\,\delta\psi^\dagger\bar\sigma^\mu\partial_\mu\psi + i\psi^\dagger\bar\sigma^\mu\partial_\mu(\delta\psi) - \partial_\mu(\delta\bar\phi)\partial^\mu\phi - \partial_\mu\bar\phi\,\partial^\mu(\delta\phi)$$

$$= i(i\,\partial_\nu\bar\phi\,\varepsilon\sigma^\nu)\bar\sigma^\mu\partial_\mu\psi - \partial_\mu\bar\phi\,\varepsilon\,\partial^\mu\psi + h.c.$$

1st term should be (+)

$$\delta S = \int d^4x\,\left(-\partial_\nu\bar\phi\,\varepsilon\sigma^\nu\bar\sigma^\mu\partial_\mu\psi - \partial_\mu\bar\phi\,\varepsilon\,\partial^\mu\psi + h.c.\right)$$

$$= \int d^4x\,\left(\tfrac12\partial_\nu\bar\phi\,\varepsilon\sigma^\nu\bar\sigma^\mu\partial_\mu\psi - \tfrac12\partial_\mu\bar\phi\,\varepsilon\sigma^\mu\bar\sigma^\nu\partial_\nu\psi - \partial_\mu\bar\phi\,\varepsilon\,\partial^\mu\psi + h.c.\right)$$

$$= \int d^4x\,\left(\tfrac12\partial_\mu\bar\phi\,\varepsilon\sigma^\nu\bar\sigma^\mu\partial_\nu\psi - \tfrac12\partial_\mu\bar\phi\,\varepsilon\sigma^\mu\bar\sigma^\nu\partial_\nu\psi - \partial_\mu\bar\phi\,\varepsilon\,\partial^\mu\psi + h.c.\right)$$

$$= \int d^4x\,\left(\partial_\mu\bar\phi\,\varepsilon\left(\tfrac12\sigma^\nu\bar\sigma^\mu + \tfrac12\sigma^\mu\bar\sigma^\nu\right)\partial_\nu\psi - \partial_\mu\bar\phi\,\varepsilon\,\partial^\mu\psi + h.c.\right)$$

$$= \int d^4x\,\left(\partial_\mu\bar\phi\,\varepsilon\,\partial^\mu\psi - \partial_\mu\bar\phi\,\varepsilon\,\partial^\mu\psi + h.c.\right) = 0$$

4.2) Calculate $[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]$ by acting with it on the fields. You should find that the combination of 2 supersymmetry transformations is a spacetime translation.

$$\varepsilon_2^a\,\delta_1\psi_a = -i\left(\varepsilon_2^a\sigma^\mu_{a\dot b}\,\varepsilon_1^{\dagger\dot b}\right)\partial_\mu\phi$$

$$\delta_{\varepsilon_1}\delta_{\varepsilon_2}\phi = \delta_{\varepsilon_1}\varepsilon_2\psi = -i\varepsilon_2\sigma^\mu\varepsilon_1^\dagger\partial_\mu\phi$$

$$\delta_{\varepsilon_2}\delta_{\varepsilon_1}\phi = -i\varepsilon_1\sigma^\mu\varepsilon_2^\dagger\partial_\mu\phi$$

$$\Rightarrow [\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\phi = -i(\varepsilon_2\sigma^\mu\varepsilon_1^\dagger - \varepsilon_1\sigma^\mu\varepsilon_2^\dagger)\partial_\mu\phi$$

$$\delta_{\varepsilon_1}\delta_{\varepsilon_2}\bar\phi = \delta_{\varepsilon_1}\varepsilon_2^\dagger\psi^\dagger = -i\varepsilon_1\partial_\mu\bar\phi\,\sigma^\mu\varepsilon_2^\dagger$$

$$\Rightarrow [\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\bar\phi = -i(\varepsilon_1\sigma^\mu\varepsilon_2^\dagger - \varepsilon_2\sigma^\mu\varepsilon_1^\dagger)\partial_\mu\bar\phi$$

$$\varepsilon^\dagger_{2\dot a}\,\delta_1\psi^{\dagger\dot a} = i\varepsilon_2^{\dot a}\,\partial_\mu\bar\phi\,\varepsilon_1^b\,\sigma^\mu_{b\dot a}$$

$$= -i\varepsilon_1^b\sigma^\mu_{b\dot a}\varepsilon_2^{\dot a}\,\partial_\mu\bar\phi$$

? shouldn't be there I think

$$\delta_{\varepsilon_1}\delta_{\varepsilon_2}\psi = -i\delta_{\varepsilon_1}\sigma^\mu\varepsilon_2^\dagger\partial_\mu\phi = -i\varepsilon_1\sigma^\mu\varepsilon_2^\dagger\partial_\mu\psi$$

$$\Rightarrow [\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\psi = -i(\varepsilon_1\sigma^\mu\varepsilon_2^\dagger - \varepsilon_2\sigma^\mu\varepsilon_1^\dagger)\partial_\mu\psi$$

$$\delta_{\varepsilon_1}\delta_{\varepsilon_2}\psi^\dagger = i\,\delta_{\varepsilon_1}\partial_\mu\bar\phi\,\varepsilon_2\sigma^\mu = i\,\varepsilon_1^\dagger\partial_\mu\psi^\dagger\varepsilon_2\sigma^\mu$$

$$\Rightarrow [\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\psi^\dagger = -i\left(\varepsilon_2\sigma^\mu\varepsilon_1^\dagger - \varepsilon_1\sigma^\mu\varepsilon_2^\dagger\right)\partial_\mu\psi^\dagger$$

4.3) Show that $\{|Q]_a, \langle Q^\dagger|_{\dot b}\} = p_{a\dot b}\times$ (sum of number ops).

$$\{|Q]_a,\langle Q^\dagger|_{\dot b}\} = \int d\tilde p_1\, d\tilde p_2\, |p_1]_a\langle p_2|_{\dot b}\{a_+(p_1)b_+^\dagger(p_1) - b_-(p_1)a_-^\dagger(p_1),\; a_-(p_2)b_-^\dagger(p_2) - b_+(p_2)a_+^\dagger(p_2)\}$$

if $\{B,\cdot\} = \{\cdot, C\} = 0$:

$\{AB, CD\} = \{CD, AB\}$

assuming $\{A,\cdot\} = 0 = \{\cdot, D\}$

$$\{AB, CD\} = A\cancel{[B,CD]}^{=0} + \{A,CD\}B$$
$$= \cancel{\{A,C\}}^{=0}DB - C[A,D]B$$

$$= \int d\tilde p_1\, d\tilde p_2\, |p_1]_a\langle p_2|_{\dot b}\Big(\{a_+(p_1)b_+^\dagger(p_1), -b_+(p_2)a_+^\dagger(p_2)\}$$
$$+\{-b_-(p_1)a_-^\dagger(p_1), a_-(p_2)b_-^\dagger(p_2)\}\Big)$$

$$= \int d\tilde p_1\, d\tilde p_2\, |p_1]_a\langle p_2|_{\dot b}\Big(b_+(p_2)(2\pi)^3 2E_{p_1}\delta^3(p_1-p_2)b_+^\dagger(p_1)$$
$$+ \begin{smallmatrix} +\leftrightarrow - \\ 1\leftrightarrow 2\end{smallmatrix}\text{ version}\Big)$$

$$= \int d\tilde p\; p_{a\dot b}\sum_s b_s(p)b_s^\dagger(p) = -\int d\tilde p\; p_{a\dot b}\sum_s\Big(\underbrace{b_s^\dagger(p)b_s(p)}_{\text{number operator}} + (2\pi)^3 2E_p\,\delta^3(0)\Big)$$

ignore

4.4) Consider the interaction terms $\mathcal{L}_I = \frac{1}{2}g\phi\psi\psi + \frac{1}{2}g^*\bar\phi\psi^\dagger\psi^\dagger - \frac{1}{4}|g|^2|\phi|^4$.

Show that the new theory $\mathcal{L}_0 + \mathcal{L}_I$ is invariant under

$$\begin{cases} \delta_\varepsilon\phi = \varepsilon\psi, \qquad \delta_\varepsilon\bar\phi = \varepsilon^\dagger\psi^\dagger \\ \delta_\varepsilon\psi_a = -i(\sigma^\mu)_{a\dot b}(\varepsilon^\dagger)^{\dot b}\partial_\mu\phi + \frac{1}{2}g^*\bar\phi^2\varepsilon_a \\ \delta_\varepsilon(\psi^\dagger)_{\dot a} = i\partial_\mu\bar\phi\,\varepsilon^b(\sigma^\mu)_{b\dot a} + \frac{1}{2}g\phi^2(\varepsilon^\dagger)_{\dot a}\end{cases}$$

still cancels in δS

$$\delta\mathcal{L}_0 = i\,\delta\psi^\dagger\bar\sigma^\mu\partial_\mu\psi - \partial_\mu\bar\phi\,\partial^\mu(\delta\phi) + h.c.$$

$$= i(i\partial_\nu\bar\phi\,\varepsilon\sigma^\nu)\bar\sigma^\mu\partial_\mu\psi + i(\tfrac{1}{2}g\phi^2\varepsilon^\dagger)\bar\sigma^\mu\partial_\mu\psi - \partial_\mu\bar\phi\,\varepsilon\partial^\mu\psi + h.c.$$

$$\delta\mathcal{L}_I = \tfrac{1}{2}g(\delta\phi)\psi\psi + g\phi(\delta\psi)\psi - \tfrac{1}{2}|g|^2|\phi|^2\bar\phi\,\delta\phi + h.c.$$

$$= \frac{1}{2} g \epsilon \psi \psi \psi - i g \phi \psi \sigma^\mu \epsilon^\dagger \partial_\mu \phi + \frac{1}{2} |g|^2 \cancel{|\phi|^2 \bar{\phi}} \epsilon \psi - \frac{1}{2} |g|^2 \cancel{|\phi|^2 \bar{\phi}} \epsilon \psi + h.c.$$

Integrate by parts in δS_0:

$$\int d^4x \, \frac{i}{2} g \phi^2 \epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \psi = -\int d^4x \, i g \phi \partial_\mu \phi \, \epsilon^\dagger \bar{\sigma}^\mu \psi = -\int d^4x \, i g \phi \partial_\mu \phi \, \psi \sigma^\mu \epsilon^\dagger$$

cancels in δS_I.

need to redo this using $\phi, \bar{\phi}$ e.o.m. to cancel

4.5) Consider the susy Ward identities

① $A_4(\phi\phi\bar{\phi}\bar{\phi}) = \frac{\langle 43 \rangle}{\langle 42 \rangle} A_4(\phi f^- f^+ \bar{\phi})$

② $A_4(\phi f^- \bar{\phi} f^+) = -\frac{\langle 23 \rangle}{\langle 24 \rangle} A_4(\phi f^- f^+ \bar{\phi})$

③ $|2\rangle A_4(\phi\phi\bar{\phi}\bar{\phi}) - |3\rangle A_4(\phi f^- f^+ \bar{\phi}) - |4\rangle A_4(\phi f^- \bar{\phi} f^+) = 0$

Plug ① & ② into ③ to show that there is no new info in ③.

$$\underbrace{\left(|2\rangle \frac{\langle 43 \rangle}{\langle 42 \rangle} - |3\rangle + |4\rangle \frac{\langle 23 \rangle}{\langle 24 \rangle} \right)}_{\times \langle 42 \rangle} A_4(\phi f^- f^+ \bar{\phi})$$

$$|2\rangle\langle 43 \rangle - |3\rangle\langle 42 \rangle - |4\rangle\langle 23 \rangle = -\left(|2\rangle\langle 34 \rangle + |3\rangle\langle 42 \rangle + |4\rangle\langle 23 \rangle \right) = 0 \quad \text{(Schouten)}$$

4.6) Derive $A_4(f^- f^- f^+ f^+) = -\frac{\langle 12 \rangle}{\langle 24 \rangle} A_4(\phi f^- f^+ \bar{\phi})$ as a susy Ward identity.

$[Q^\dagger, b_-] = |p\rangle a_-$

$[Q^\dagger, a_+] = |p\rangle b_+$

$$0 = \langle 0 | [Q^\dagger, b_-(1) b_-(2) b_+(3) a_+(4)] | 0 \rangle$$

$$= |1\rangle \langle a_- b_- b_+ a_+ \rangle - |2\rangle \langle b_- a_- b_+ a_+ \rangle + |4\rangle \langle b_- b_- b_+ b_+ \rangle$$

$$= |1\rangle A_4(\phi f^- f^+ \bar{\phi}) - |2\rangle A_4(f^- \phi f^+ \bar{\phi}) + |4\rangle A_4(f^- f^- f^+ f^+)$$

$$\Rightarrow -\langle 12 \rangle A_4(\phi f^- f^+ \bar{\phi}) = \langle 24 \rangle A_4(f^- f^- f^+ f^+)$$

$$\Rightarrow A_4(f^- f^- f^+ f^+) = \frac{-\langle 12 \rangle}{\langle 24 \rangle} A_4(\phi f^- f^+ \bar{\phi})$$

4.7) Find a Q-susy Ward identity that gives

$$A_4(\phi f^- f^+ \bar{\phi}) = -\frac{[13]}{[12]} A_4(\phi\phi\bar{\phi}\bar{\phi})$$

Show that this is equivalent to $A_4(\phi f^- f^+ \bar{\phi}) = \frac{\langle 24 \rangle}{\langle 34 \rangle} A_4(\phi\phi\bar{\phi}\bar{\phi})$.

$[Q, a_-] = |p]\, b_-$
$[Q, b_+] = |p]\, a_+$

$$0 = \langle 0 | [Q, a_-(1) a_-(2) b_+(3) a_+(4)] | 0 \rangle$$
$$= |1] \langle b_- a_- b_+ a_+ \rangle + |2] \langle a_- b_- b_+ a_+ \rangle + |3] \langle a_- a_- a_+ a_+ \rangle$$
$$= |1] A_4(f^- \phi f^+ \phi) + |2] A_4(\phi f^- f^+ \bar{\phi}) + |3] A_4(\phi \phi \bar{\phi} \bar{\phi})$$
$$\Rightarrow A_4(\phi f^- f^+ \bar{\phi}) = -\frac{[13]}{[12]} A_4(\phi \phi \bar{\phi} \bar{\phi})$$
$$-\frac{[13]}{[12]} = \frac{-[13]\langle 12 \rangle}{[34]\langle 34 \rangle} = \frac{[43]\langle 42 \rangle}{[34]\langle 34 \rangle} = \frac{\langle 24 \rangle}{\langle 34 \rangle} \Rightarrow A_4(\phi f^- f^+ \bar{\phi}) = \frac{\langle 24 \rangle}{\langle 34 \rangle} A_4(\phi \phi \bar{\phi} \bar{\phi}).$$

4.8) The non-vanishing of the $n=3$ anti-MHV amplitude $A_3[1^- 2^+ 3^+]$ escapes the Ward identity that forces $A_n[g^- g^+ g^+ \cdots g^+] = 0$ for $n > 3$. Explain how.

To derive the Ward identity for general n:

$$\langle 0 | [\tilde{Q}, b_{1-} c_{2+} c_{3+} c_{4+} \cdots] | 0 \rangle = |1\rangle \langle c_{1-} c_{2+} c_{3+} c_{4+} \cdots \rangle = |1\rangle A_n[- + + \cdots +] = 0$$

For $n=3$ we have

$$\langle 0 | [\tilde{Q}, b_{1-} c_{2+} c_{3+}] | 0 \rangle = |1\rangle A_3[- + +] = 0$$

But little gp scaling fixes $A_3[- + +] \sim \frac{[23]^3}{[12][13]}$ so from 3-particle special kinematics all $\langle\,\rangle$ brackets are 0. so the Ward identity is satisfied by $\langle 12 \rangle = \langle 13 \rangle = 0$ & doesn't tell us $A_3 = 0$.

4.9) Show in $\mathcal{N}=1$ SYM that supersymmetry Ward identities give

$$A_n[g^- \lambda^- \lambda^+ g^+ \cdots g^+] = \frac{\langle 13 \rangle}{\langle 12 \rangle} A_n[g^- g^- g^+ g^+ \cdots g^+].$$

$$\langle 0 | [\tilde{Q}, c_- c_- b_+ c_+ \cdots] | 0 \rangle = |1\rangle A_n[\lambda^- g^- \lambda^+ g^+ \cdots] + |2\rangle A_n[g^- \lambda^- \lambda^+ g^+ \cdots]$$
$$+ |3\rangle A_n[g^- g^- g^+ g^+ \cdots]$$
$$\Rightarrow A_n[g^- \lambda^- \lambda^+ g^+ \cdots g^+] = \frac{\langle 13 \rangle}{\langle 12 \rangle} A_n[g^- g^- g^+ \cdots g^+]$$

Relate $A_n[\lambda^-\lambda^-\lambda^+\lambda^+g^+\cdots g^+]$ to the MHV gluon amplitude.

$$\langle 0|[\tilde{Q}, c_- b_- b_+ b_+ c_+ \cdots]|0\rangle = |1\rangle A_n[\lambda^-\lambda^-\lambda^+\lambda^+g^+\cdots]$$
$$-|3\rangle A_n[g^-\lambda^-g^+\lambda^+g^+\cdots] + |4\rangle A_n[g^-\lambda^-\lambda^+g^+g^+\cdots]$$

$$\Rightarrow A_n[g^-\lambda^-\lambda^+g^+\cdots] = \frac{\langle 13\rangle}{\langle 34\rangle} A_n[\lambda^-\lambda^-\lambda^+\lambda^+g^+\cdots]$$

$$\Rightarrow A_n[\lambda^-\lambda^-\lambda^+\lambda^+g^+\cdots g^+] = \frac{\langle 34\rangle}{\langle 12\rangle} A_n[g^-g^-g^+\cdots g^+]$$

Does $\mathcal{N}=1$ susy allow you to relate gluon amplitudes such as $A_n[1^-2^-3^+\cdots n^+]$ & $A_n[1^-2^+3^-\cdots n^+]$?

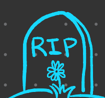
$$\updownarrow \tilde{Q}$$
$$[g^-\lambda^-\lambda^+\cdots] \overset{?}{\Leftrightarrow} [g^-g^+g^-\cdots]$$

No b/c we don't have ops that can lower by helicity $=\pm 1$ (?)

$$[Q^2, c_+] = Q[Q,c_+] - [Q,c_+]Q$$
$$= [i|(Qb_- - b_-Q) = [i|[Q,b_-]$$
$$= [i|\,[i|\,c_-$$

Can it relate gluon MHV & NMHV amplitudes?

No, for the same reason as above.



$$[\tilde{Q}^\dagger, c_+] = |i]\,b_+$$
$$[\tilde{Q}^\dagger, b_-] = |i]\,c_-$$
$$[Q^\dagger, b_-] = \langle i|\,c_-$$
$$[Q^\dagger, c_-] = \langle i|\,b_-$$

4.10) Show that the susy Ward identities give the following relationship among color-ordered amplitudes in $\mathcal{N}=4$ SYM:

$$0 = -|1\rangle A_n[\lambda^{123}g^-\lambda^4g^+\cdots g^+] - |2\rangle A_n[g^-\lambda^{123}\lambda^4g^+\cdots g^+]$$
$$+ |3\rangle A_n[g^-g^-g^+\cdots g^+]$$

In particular, $A_n[g^-\lambda^{123}\lambda^4g^+\cdots g^+] = \frac{\langle 13\rangle}{\langle 12\rangle} A_n[g^-g^-g^+\cdots g^+]$.

$$\langle 0|[\tilde{Q}_4, a^{1234} a^{1234} a^4 a \cdots a]|0\rangle = -|1\rangle \overset{?}{6} A_n[\lambda^{123} g^- \lambda^4 g^4 \cdots]$$

$$-|2\rangle \overset{?}{6} A_n[g^- \lambda^{123} \lambda^4 g^+ \cdots] + |3\rangle A_n[g^- g^- g^+ g^+ \cdots]$$

$$\delta_4^{[1} a^{234]} = \frac{1}{4!}(-a^{123} - a^{231} - a^{312} + a^{132} + a^{213} + a^{321}) \overset{?}{=} 6a^{123}$$

Then derive $A_n[g^- s^{12} s^{34} g^+ \cdots g^+] = \frac{\langle 13\rangle^2}{\langle 12\rangle^2} A_n[g^- g^- g^+ g^+ \cdots g^+]$

$$\langle 0|[\tilde{Q}_3, a^{1234} a^{123} a^{34} a \cdots a]|0\rangle = -|1\rangle \overset{?}{6} A_n[\lambda^{124} \lambda^{123} s^{34} g^+ \cdots]$$

$$+|2\rangle \overset{?}{2} A_n[g^- s^{12} s^{34} g^+ \cdots] - |3\rangle A_n[g^- \lambda^{123} \lambda^4 g^+ \cdots]$$

$$\Rightarrow A_n[g^- \lambda^{123} \lambda^4 g^+ \cdots g^+] = \frac{\langle 12\rangle}{\langle 13\rangle} A_n[g^- s^{12} s^{34} g^+ \cdots g^+]$$

$$\Rightarrow A_n[g^- s^{12} s^{34} g^+ \cdots g^+] = \frac{\langle 13\rangle^2}{\langle 12\rangle^2} A_n[g^- g^- g^+ g^+ \cdots g^+]$$

$$[Q_3, a^{1234}]$$

$$[Q_3, a^{123}] = |p\rangle(a^{12} - a^{21}) \overset{?}{=} |p\rangle 2a^{12}$$

4.11) Derive $A_n[g^+ g^+ \cdots g_i^- \cdots g_j^- \cdots g^+] = \frac{\langle ij\rangle^4}{\langle 12\rangle^4} A_n[g^- g^- g^+ \cdots g^+]$ by extending the strategy you used in (4.10).

Strategy: 1. change g^- to λ^{123} using $\tilde{Q}_4$
2. change λ^{123} to s^{23} using $\tilde{Q}_1$
3. change s^{23} to λ^3 using $\tilde{Q}_2$
4. change λ^3 to g^+ using $\tilde{Q}_3$

$\uparrow$ go other way w/ Q for $g^+ \to g^-$

$$\langle 0|[\tilde{Q}_4, a_1^4 a_2 \cdots a_i^{1234} \cdots a_j^{1234} \cdots a_n]|0\rangle = |1\rangle A_n[g^+ g^+ \cdots g_i^- \cdots g_j^- \cdots g^+]$$

$$+|i\rangle A_n[\lambda^4 g^+ \cdots \lambda_i^{123} \cdots g_j^- \cdots g^+] - |j\rangle A_n[\lambda^4 g^+ \cdots g_i^- \cdots \lambda_j^{123} \cdots g^+]$$

$$\Rightarrow A_n[g^+ g^+ \cdots g_i^- \cdots g_j^- \cdots g^+] = -\frac{\langle ij\rangle}{\langle 1j\rangle} A_n[\lambda^4 g^+ \cdots \lambda_i^{123} \cdots g_j^- \cdots g^+]$$

$$\langle 0|[\tilde{Q}_3, a_1^{34} a_2 \cdots a_i^{312} \cdots a_j^{1234} \cdots a_n]|0\rangle = |1\rangle A_n[\lambda^4 g^+ \cdots \lambda_i^{123} \cdots g_j \cdots g^+]$$

$$+|i\rangle A_n[s^{34} g^+ \cdots s_i^{12} \cdots g_j^- \cdots g^+] - |j\rangle A_n[s^{34} g^+ \cdots \lambda_i^{123} \cdots \lambda_j^{124} \cdots g^+]$$

$$\Rightarrow A_n[\lambda^4 g^+ \cdots \lambda_i^{123} \cdots g_j^- \cdots g^+] = -\frac{\langle ij\rangle}{\langle 1j\rangle} A_n[s^{34} g^+ \cdots s_i^{12} \cdots g_j^- \cdots g^+]$$

$$\langle 0|[\tilde{Q}_2, a_1^{234} a_2 \cdots a_i^{21} \cdots a_j^{1234} \cdots a_n]|0\rangle = -|1\rangle \mathcal{A}_n[S^{34} g^+ \cdots S_i^{12} \cdots g_j^- \cdots g^+]$$

$$-|i\rangle \mathcal{A}_n[\lambda^{234} g^+ \cdots \lambda_i^1 \cdots g_j^- \cdots g^+] - |j\rangle \mathcal{A}_n[\lambda^{234} g^+ \cdots S_i^{21} \cdots \lambda_j^{134} \cdots g^+]$$

$$\Rightarrow \mathcal{A}_n[S^{34} g^+ \cdots S_i^{12} \cdots g_j^- \cdots g^+] = -\frac{\langle ij\rangle}{\langle 1j\rangle} \mathcal{A}_n[\lambda^{234} g^+ \cdots \lambda_i^1 \cdots g_j^- \cdots g^+]$$

$$\langle 0|[\tilde{Q}_1, a_1^{1234} a_2 \cdots a_i^1 \cdots a_j^{1234} \cdots a_n]|0\rangle = |1\rangle \mathcal{A}_n[\lambda^{234} g^+ \cdots \lambda_i^1 \cdots g_j^- \cdots g^+]$$

$$+|i\rangle \mathcal{A}_n[g^- g^+ \cdots g_i^+ \cdots g_j^- \cdots g^+] - |j\rangle \mathcal{A}_n[g^- g^+ \cdots \lambda_i^1 \cdots \lambda_j^{234} \cdots g^+]$$

$$\Rightarrow \mathcal{A}_n[\lambda^{234} g^+ \cdots \lambda_i^1 \cdots g_j^- \cdots g^+] = -\frac{\langle ij\rangle}{\langle 1j\rangle} \mathcal{A}_n[g^- g^+ \cdots g_i^+ \cdots g_j^- \cdots g^+]$$

$$\Rightarrow \mathcal{A}_n[g^+ g^+ \cdots g_i^- \cdots g_j^- \cdots g^+] = \left(-\frac{\langle ij\rangle}{\langle 1j\rangle}\right)^4 \mathcal{A}_n[g^- g^+ \cdots g_i^+ \cdots g_j^- \cdots g^+]$$

repeat for g_2^+, g_j^-:

$$\langle 0|[\tilde{Q}_4, a_1^{1234} a_2^4 \cdots a_i \cdots a_j^{1234} \cdots a_n]|0\rangle = -|1\rangle \mathcal{A}_n[\lambda^{123} \lambda^4 \cdots g_i^+ \cdots g_j^- \cdots g^+]$$

$$+|2\rangle \mathcal{A}_n[g^- g^+ \cdots g_i^+ \cdots g_j^- \cdots g^+] + |j\rangle \mathcal{A}_n[g^- \lambda^4 \cdots g_i^+ \cdots \lambda_j^{123} \cdots g^+]$$

$$\Rightarrow \mathcal{A}_n[g^- g^+ \cdots g_i^+ \cdots g_j^- \cdots g^+] = -\frac{\langle 1j\rangle}{\langle 12\rangle} \mathcal{A}_n[g^- \lambda^4 \cdots g_i^+ \cdots \lambda_j^{123} \cdots g^+]$$

$$\langle 0|[\tilde{Q}_3, a_1^{1234} a_2^{34} \cdots a_i \cdots a_j^{312} \cdots a_n]|0\rangle = -|1\rangle \mathcal{A}_n[\lambda^{124} S^{34} \cdots g_i^+ \cdots \lambda^{312} \cdots g^+]$$

$$+|2\rangle \mathcal{A}_n[g^- \lambda^4 \cdots g_i^+ \cdots \lambda_j^{123} \cdots g^+] + |j\rangle \mathcal{A}_n[g^- S^{34} \cdots g_i^+ \cdots S^{12} \cdots g^+]$$

$$\Rightarrow \mathcal{A}_n[g^- \lambda^4 \cdots g_i^+ \cdots \lambda_j^{123} \cdots g^+] = -\frac{\langle 1j\rangle}{\langle 12\rangle} \mathcal{A}_n[g^- S^{34} \cdots g_i^+ \cdots S^{12} \cdots g^+]$$

$$\langle 0|[\tilde{Q}_2, a_1^{1234} a_2^{234} \cdots a_i \cdots a_j^{21} \cdots a_n]|0\rangle = -|1\rangle \mathcal{A}_n[\lambda^{134} \lambda^{234} \cdots g_i^+ \cdots S_j^{21} \cdots g^+]$$

$$+|2\rangle \mathcal{A}_n[g^- S^{34} \cdots g_i^+ \cdots S^{12} \cdots g^+] + |j\rangle \mathcal{A}_n[g^- \lambda^{234} \cdots g_i^+ \cdots \lambda_j^1 \cdots g^+]$$

$$\Rightarrow \mathcal{A}_n[g^- S^{34} \cdots g_i^+ \cdots S^{12} \cdots g^+] = -\frac{\langle 1j\rangle}{\langle 12\rangle} \mathcal{A}_n[g^- \lambda^{234} \cdots g_i^+ \cdots \lambda_j^1 \cdots g^+]$$

$$\langle 0|[\tilde{Q}_1, a_1^{1234} a_2^{1234} \cdots a_i \cdots a_j^1 \cdots a_n]|0\rangle = |1\rangle \mathcal{A}_n[\lambda^{234} g^- \cdots g_i^+ \cdots \lambda_j^1 \cdots g^+]$$

$$+|2\rangle \mathcal{A}_n[g^- \lambda^{234} \cdots g_i^+ \cdots \lambda_j^1 \cdots g^+] + |j\rangle \mathcal{A}_n[g^- g^- \cdots g_i^+ \cdots g_j^+ \cdots g^+]$$

$$\Rightarrow \mathcal{A}_n[g^- \lambda^{234} \cdots g_i^+ \cdots \lambda_j^1 \cdots g^+] = -\frac{\langle 1j\rangle}{\langle 12\rangle} \mathcal{A}_n[g^- g^- \cdots g_i^+ \cdots g_j^+ \cdots g^+]$$

$$\Rightarrow A_n[g^- g^+ \cdots g_i^+ \cdots g_j^- \cdots g^+] = \left(-\frac{\langle ij\rangle}{\langle 12\rangle}\right)^4 A_n[g^- g^- \cdots g_i^+ \cdots g_j^+ \cdots g^+]$$

Finally we have

$$A_n[g^+ g^+ \cdots g_i^- \cdots g_j^- \cdots g^+] = \left(\frac{\langle ij\rangle}{\langle 1j\rangle}\right)^4 \left(\frac{\langle 1j\rangle}{\langle 12\rangle}\right)^4 A_n[g^- g^- \cdots g_i^+ \cdots g_j^+ \cdots g^+]$$

$$= \frac{\langle ij\rangle^4}{\langle 12\rangle^4} A_n[g^- g^- \cdots g_i^+ \cdots g_j^+ \cdots g^+]$$

spooky

Convince yourself that susy Ward identities cannot mix gluon amplitudes w/ different K in the N^KMHV classification.

You can only change operators in pairs w/ opposite helicity.

4.12) Show that the supercharges q^{Aa}, $\tilde{q}_A^{\dot a}$ satisfy the standard susy anti-commutation relation $\{q^{Aa}, \tilde{q}_B^{\dot b}\} = \delta^A{}_B |p\rangle^{\dot b} [p|^a = -\delta^A{}_B p^{\dot b a}$.

$$\{q^{Aa}, \tilde{q}_B^{\dot b}\}\Omega(p) = [p|^a \frac{\partial}{\partial \eta_A}\left(|p\rangle^{\dot b} \eta_B \Omega\right) + |p\rangle^{\dot b} \eta_B [p|^a \frac{\partial}{\partial \eta_A}\Omega$$

$$= p^{\dot b a}\left(\left(\frac{\partial}{\partial \eta_A}\eta_B\right)\Omega - \cancel{\eta_B \frac{\partial}{\partial \eta_A}\Omega} + \cancel{\eta_B \frac{\partial}{\partial \eta_A}\Omega}\right) = \delta^A{}_B p^{\dot b a}$$ missing minus...

The supercharges act on the spectrum by shifting states left or right in Ω. Check that this action on Ω matches the commutation relations $[Q, \cdot]$, $[\tilde{Q}, \cdot]$.

$$q^{Aa}\Omega = [p|^a \frac{\partial}{\partial \eta_A}\Omega = \lambda^A - \frac{1}{2!}\partial_A(\eta_B \eta_C) S^{BC} - \frac{1}{3!}\partial_A(\eta_B \eta_C \eta_D)\lambda^{BCD}$$
$$+ \partial_A(\eta_1 \eta_2 \eta_3 \eta_4) g^-$$

$$= [p|^a\left(\lambda^A - \eta_B S^{AB} - \frac{1}{2}\eta_B \eta_C \lambda^{ABC} + \left(\delta_A^1 \eta_2 \eta_3 \eta_4 - \delta_A^2 \eta_1 \eta_3 \eta_4\right.\right.$$
$$\left.\left. + \delta_A^3 \eta_1 \eta_2 \eta_4 - \delta_A^4 \eta_1 \eta_2 \eta_3\right) g^-\right)$$

$$\left(\begin{aligned} \partial_A(BCD) &= \partial_A B \cdot CD - B\,\partial_A(CD) \\ &= \partial_A B \cdot CD - B(\partial_A C \cdot D - C\,\partial_A D) \\ &= \delta_A{}^B CD - \delta_A{}^C BD + \delta_A{}^D BC \end{aligned}\right) \times \lambda^{BCD}$$ ... definitely doing this wrong

$$= BC\lambda^{ABC} - BC\lambda^{BAC} + BC\lambda^{BCA} = 3\eta_B \eta_C \lambda^{ABC}$$

$$\tilde{q}_A{}^{\dot a} = |p\rangle^{\dot a}\eta_A\Omega = |p\rangle^{\dot a}\left(\eta_A g^+ + \eta_A\eta_B\lambda^B - \tfrac{1}{2!}\eta_A\eta_B\eta_C S^{BC} - \tfrac{1}{3!}\eta_A\eta_B\eta_C\eta_D\lambda^{BCD}\right)$$

$$\tilde{q}_A g^+ = |p\rangle\eta_A g^+\big|_{\eta=0} = 0 \quad ??$$

how do the g^+, λ^B etc relate to a^A's?

$$\tilde{q}_A\lambda^B = |p\rangle\eta_A\eta_B\lambda^B\big|_{\eta=0} = 0 \quad ???$$

4.13) Act w/ $\hat{Q}_4$ on the superamplitude $\mathcal{A}_n^{MHV}$ to extract all terms whose Grassmann structure is $(\eta_1)^4(\eta_2)^4(\eta_{34})$. Use that to show that the "component amplitude" susy Ward identity

$$0 = -|1\rangle A_n[\lambda^{123}g^-\lambda^4 g^+\cdots g^+] - |2\rangle A_n[g^-\lambda^{123}\lambda^4 g^+\cdots g^+] + |3\rangle A_n[g^-g^-g^+\cdots g^+]$$

follows from $\tilde{Q}_4\mathcal{A}_n^{MHV} = 0$.

$$\tilde{Q}_4\mathcal{A}_n^{MHV}[\Omega_1\cdots\Omega_n] = \sum_{i=1}^{4}|i\rangle\eta_{i4}\mathcal{A}_n^{MHV}[\Omega_1\cdots\Omega_n]$$

$i=1$: want $\eta_{11}\eta_{12}\eta_{13}(\eta_2)^4\eta_{34} \Rightarrow$ need $\lambda_1^4 g_2^- \lambda_3^{123}\cdots$ in A_n

$i=2$: want $(\eta_1)^4\eta_{21}\eta_{22}\eta_{23}\eta_{34} \Rightarrow$ need $g_1^-\lambda_2^4\lambda_3^{123}\cdots$ in A_n

$i=3$: want $(\eta_1)^4(\eta_2)^4 \Rightarrow$ need $g_1^- g_2^- g^+\cdots$ in A_n

$i=4$: N/A

$$\mathcal{A}_n^{MHV} = A_n[\lambda^4 g^-\lambda^{123}g^+\cdots]\eta_{11}\eta_{12}\eta_{13}(\eta_2)^4\eta_{34} + A_n[g^-\lambda^4\lambda^{123}g^+\cdots](\eta_1)^4\eta_{21}\eta_{22}\eta_{23}\eta_{34} + A_n[g^-g^-g^+g^+\cdots](\eta_1)^4(\eta_2)^4 + \ldots$$

$$\tilde{Q}_4\mathcal{A}_n^{MHV} = \left(-|1\rangle A_n[\lambda^4 g^-\lambda^{123}g^+\cdots] - |2\rangle A_n[g^-\lambda^4\lambda^{123}g^+\cdots] + |3\rangle A_n[g^-g^-g^+g^+\cdots]\right) \times (\eta_1)^4(\eta_2)^4\eta_{34}$$

$$\Rightarrow -|1\rangle A_n[\lambda^4 g^-\lambda^{123}g^+\cdots] - |2\rangle A_n[g^-\lambda^4\lambda^{123}g^+\cdots] + |3\rangle A_n[g^-g^-g^+g^+\cdots] = 0$$

4.14) Show that momentum conservation ensures that Q^A annihilates $\delta^{(8)}(\tilde{Q})$.

where $\delta^{(8)}(\tilde{Q}) = \frac{1}{2^4}\prod_{A=1}^{4}\tilde{Q}_{A\dot a}\tilde{Q}_A^{\dot a} = \frac{1}{2^4}\prod_{A=1}^{4}\prod_{i,j=1}^{n}\langle ij\rangle\eta_{iA}\eta_{jA}$.

$$Q^A \delta^{(8)}(\tilde{Q}) \sim \sum_{k=1}^{n} [k| \frac{\partial}{\partial \eta_{kA}} \prod_{B=1}^{4} \sum_{i,j=1}^{n} \langle ij\rangle \eta_{iB}\eta_{jB}$$

$$= \sum_{k=1}^{n} [k| (-1)^{\#} \left(\prod_{B\neq A} \sum_{i,j=1}^{n} \langle ij\rangle \eta_{iB}\eta_{jB} \right) \left(\sum_{i,j=1}^{n} \langle ij\rangle \frac{\partial}{\partial \eta_{kA}} (\eta_{iA}\eta_{jA}) \right)$$

$$\sim \sum_{i,j,k}^{n} [k| \langle ij\rangle (\delta_{ik}\eta_{jA} - \delta_{jk}\eta_{iA}) = \sum_{i,j}^{n} \langle ij\rangle ([i|\eta_{jA} - [j|\eta_{iA})$$

$$= \sum_{i,j}^{n} (-\langle j \cancelto{0}{i\rangle[i|} \eta_{jA} - \langle i \cancelto{0}{j\rangle[j|} \eta_{iA}) \qquad \text{from} \quad \sum_{i=1}^{n} |i\rangle[i| = \sum_{j=1}^{n} |j\rangle[j| = 0$$

Which property of angle spinors allows you to demonstrate explicitly that $\tilde{Q}_A \delta^{(8)}(\tilde{Q}) = 0$?

$$\tilde{Q}_A \delta^{(8)}(\tilde{Q}) \sim \sum_{k=1}^{n} |k\rangle \eta_{kA} \prod_{B=1}^{4} \sum_{i,j=1}^{n} \langle ij\rangle \eta_{iB}\eta_{jB}$$

$$= \sum_{k=1}^{n} |k\rangle (-1)^{\#} \left(\prod_{B\neq A} \sum_{i,j=1}^{n} \langle ij\rangle \eta_{iB}\eta_{jB} \right) \left(\sum_{i,j=1}^{n} \langle ij\rangle \eta_{kA}\eta_{iA}\eta_{jA} \right)$$

$$\sim \sum_{i,j,k}^{n} |k\rangle\langle ij\rangle \eta_{kA}\eta_{iA}\eta_{jA} = \frac{1}{3} \sum_{i,j,k} \left(\cancelto{0}{|k\rangle\langle ij\rangle + |i\rangle\langle jk\rangle + |j\rangle\langle ki\rangle} \right) \eta_{kA}\eta_{iA}\eta_{jA}$$

$= 0$ by Schouten identity

4.15) Reproduce the susy Ward identities

$$A_n[g^- \lambda^{123} \lambda^4 g^+ \cdots g^+] = \frac{\langle 13\rangle}{\langle 12\rangle} A_n[g^- g^- g^+ \cdots g^+].$$

$$A_n[g^- S^{12} S^{34} g^+ \cdots g^+] = \frac{\langle 13\rangle^2}{\langle 12\rangle^2} A_n[g^- g^- g^+ g^+ \cdots g^+]$$

$$A_n[g^+ g^+ \cdots g_i^- \cdots g_j^- \cdots g^+] = \frac{\langle ij\rangle^4}{\langle 12\rangle^4} A_n[g^- g^- g^+ g^+ \cdots g^+]$$

using the MHV superamplitude $\mathcal{A}_n^{MHV}[123\cdots n] = \dfrac{\delta^{(8)}(\tilde{Q})}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle}$

$$\left(\frac{\partial}{\partial \eta_1}\right)^4 \left(\frac{\partial}{\partial \eta_{21}} \frac{\partial}{\partial \eta_{22}} \frac{\partial}{\partial \eta_{23}} \right) \left(\frac{\partial}{\partial \eta_{34}} \right) \mathcal{A}_n^{MHV}[123\cdots n] = A_n[g^- \lambda^{123} \lambda^4 g^+ \cdots g^+]$$

$$= \frac{1}{\langle 12\rangle \cdots \langle n1\rangle} \left(\frac{\partial}{\partial \eta_{21}} \frac{\partial}{\partial \eta_{22}} \frac{\partial}{\partial \eta_{23}} \right) \left(\frac{\partial}{\partial \eta_{34}} \right) \prod_{A=1}^{4} \sum_{j\neq 1} \langle 1j\rangle \eta_{jA}$$

$$= \frac{1}{\langle 12\rangle \cdots \langle n1\rangle} \left(\prod_{A=1}^{3} \frac{\partial}{\partial \eta_{2A}} \sum_{j\neq 1} \langle 1j\rangle \eta_{jA} \right) \left(\frac{\partial}{\partial \eta_{34}} \sum_{j\neq 1} \langle 1j\rangle \eta_{j4} \right)$$

$$= \frac{1}{\langle12\rangle\cdots\langle n1\rangle}\left(\prod_{A=1}^{3}\langle12\rangle\right)\langle13\rangle = \frac{\langle12\rangle^3\langle13\rangle}{\langle12\rangle\cdots\langle n1\rangle} = \frac{\langle13\rangle}{\langle12\rangle} A_n[g^-g^-g^+\cdots g^+]$$

I feel like this isn't how it should be done...

$$\left(\frac{\partial}{\partial\eta_1}\right)^4\left(\frac{\partial}{\partial\eta_{21}}\frac{\partial}{\partial\eta_{22}}\right)\left(\frac{\partial}{\partial\eta_{33}}\frac{\partial}{\partial\eta_{34}}\right)\mathcal{A}_n^{MHV}[123\cdots n] = A_n[g^-S^{12}S^{34}g^+\cdots g^+]$$

$$= \frac{1}{\langle12\rangle\cdots\langle n1\rangle}\left(\prod_{A=1}^{2}\frac{\partial}{\partial\eta_{2A}}\sum_{j\neq1}\langle1j\rangle\eta_{jA}\right)\left(\prod_{A=3}^{4}\frac{\partial}{\partial\eta_{3A}}\sum_{j\neq1}\langle1j\rangle\eta_{jA}\right) = \frac{\langle12\rangle^2\langle13\rangle^2}{\langle12\rangle\cdots\langle n1\rangle}$$

$$= \frac{\langle13\rangle^2}{\langle12\rangle^2}A_n[g^-g^-g^+\cdots g^+]$$

$$\left(\frac{\partial}{\partial\eta_i}\right)^4\left(\frac{\partial}{\partial\eta_j}\right)^4\mathcal{A}_n^{MHV}[123\cdots n] = A_n[g^+\cdots g_i^-\cdots g_j^-\cdots g^+]$$

$$= \frac{1}{\langle12\rangle\cdots\langle n1\rangle}\left(\frac{\partial}{\partial\eta_j}\right)^4\prod_{A=1}^{4}\sum_{k\neq i}\langle ik\rangle\eta_{kA} = \frac{1}{\langle12\rangle\cdots\langle n1\rangle}\prod_{A=1}^{4}\frac{\partial}{\partial\eta_{jA}}\sum_{k\neq i}\langle ik\rangle\eta_{kA}$$

$$= \frac{\langle ij\rangle^4}{\langle12\rangle\cdots\langle n1\rangle} = \frac{\langle ij\rangle^4}{\langle12\rangle^4}A_n[g^-g^-g^+\cdots g^+]$$

4.16) Use the $\mathcal{N}=4$ SYM superamplitude to calculate the 4-scalar amplitude $A_4[S^{12}S^{34}S^{12}S^{34}]$. Compare your answer to the non-susy 4-scalar amplitude

$$A_4(\phi\phi^*\phi\phi^*) = \tilde{e}^2\frac{\langle13\rangle^2\langle24\rangle^2}{\langle12\rangle\langle23\rangle\langle34\rangle\langle41\rangle}.$$

$$A_4[S^{12}S^{34}S^{12}S^{34}] = \frac{\partial}{\partial\eta_{11}}\frac{\partial}{\partial\eta_{12}}\frac{\partial}{\partial\eta_{23}}\frac{\partial}{\partial\eta_{24}}\frac{\partial}{\partial\eta_{31}}\frac{\partial}{\partial\eta_{32}}\frac{\partial}{\partial\eta_{43}}\frac{\partial}{\partial\eta_{44}}\mathcal{A}_4[1234]$$

$$= \frac{1}{\langle12\rangle\cdots\langle41\rangle}\left(\prod_{A=1}^{2}\frac{\partial}{\partial\eta_{1A}}\sum_{j\neq3}\langle3j\rangle\eta_{jA}\right)\left(\prod_{A=3}^{4}\frac{\partial}{\partial\eta_{2A}}\sum_{j\neq4}\langle4j\rangle\eta_{jA}\right)$$

$$= \frac{\langle13\rangle^2\langle24\rangle^2}{\langle12\rangle\langle23\rangle\langle34\rangle\langle41\rangle} \quad \text{same as } A_4(\phi\phi^*\phi\phi^*) \text{ w/o } \tilde{e}^2$$

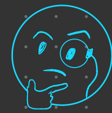
Calculate $A_4[S^{12}S^{23}S^{34}S^{41}]$.

$$A_4[S^{12}S^{23}S^{34}S^{41}] = \frac{1}{\langle12\rangle\cdots\langle41\rangle}\left(\frac{\partial}{\partial\eta_{41}}\sum_{j\neq1}\langle1j\rangle\eta_{j1}\right)\left(\frac{\partial}{\partial\eta_{12}}\sum_{j\neq2}\langle2j\rangle\eta_{j2}\right)$$

$$\times\left(\frac{\partial}{\partial\eta_{23}}\sum_{j\neq3}\langle3j\rangle\eta_{j3}\right)\left(\frac{\partial}{\partial\eta_{34}}\sum_{j\neq4}\langle4j\rangle\eta_{j4}\right)$$

$$= \frac{\langle 14\rangle\langle 21\rangle\langle 32\rangle\langle 43\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} = 1$$

should be $\mathcal{A}_3$?



4.17) Show that $Q^A \mathcal{A}_n^{\text{anti-MHV}} = \tilde{Q}_A \mathcal{A}_n^{\text{anti-MHV}} = 0$.

$$\mathcal{A}_3^{\text{anti-MHV}} = \frac{1}{[12][23][31]} \prod_{A=1}^{4} \left([12]\eta_{3A} + [23]\eta_{1A} + [31]\eta_{2A}\right)$$

$$Q^A \mathcal{A}_3^{\text{anti-MHV}} = \frac{1}{[12][23][31]} \sum_{i=1}^{3} [i| \frac{\partial}{\partial \eta_{iA}} \prod_{B=1}^{4} \left([12]\eta_{3B} + [23]\eta_{1B} + [31]\eta_{2B}\right)$$

$$= \frac{1}{[12][23][31]} \left(\sum_{i=1}^{3} [i| \frac{\partial}{\partial \eta_{iA}} \left([12]\eta_{3A} + [23]\eta_{1A} + [31]\eta_{2A}\right)\right)$$

$$\times \prod_{B\neq A} \left([12]\eta_{3B} + [23]\eta_{1B} + [31]\eta_{2B}\right)$$

$$= \frac{1}{[12][23][31]} \left(\cancelto{0}{[23][1| + [31][2| + [12][3|}\right) \times \prod_{B\neq A} \left([12]\eta_{3B} + [23]\eta_{1B} + [31]\eta_{2B}\right)$$

$= 0$ by Schouten

$$\tilde{Q}_A \mathcal{A}_3^{\text{anti-MHV}} = \frac{1}{[12][23][31]} \sum_{i=1}^{3} |i\rangle \eta_{iA} \prod_{B=1}^{4} \left([12]\eta_{3B} + [23]\eta_{1B} + [31]\eta_{2B}\right)$$

$$= \frac{1}{[12][23][31]} \Big(|1\rangle \eta_{1A}\left([12]\eta_{3A} + [31]\eta_{2A}\right) + |2\rangle\eta_{2A}\left([12]\eta_{3A} + [23]\eta_{1A}\right)$$

$$+ |3\rangle\eta_{3A}\left([23]\eta_{1A} + [31]\eta_{2A}\right)\Big) \times \prod_{B\neq A} \left([12]\eta_{3B} + [23]\eta_{1B} + [31]\eta_{2B}\right)$$

$$= \frac{1}{[12][23][31]} \Big(\eta_{1A}\eta_{3A}\left(|1\rangle[12] + |3\rangle[32]\right) + \eta_{2A}\eta_{1A}\left(|1\rangle[13] + |2\rangle[23]\right)$$

$$+ \eta_{3A}\eta_{2A}\left(|2\rangle[21] + |3\rangle[31]\right)\Big) \times \prod_{B\neq A} \left([12]\eta_{3B} + [23]\eta_{1B} + [31]\eta_{2B}\right)$$

$$= \eta_{1A}\eta_{3A}\left(-|2\rangle\cancelto{0}{[2|)|2]}\right) + \eta_{2A}\eta_{1A}\left(-|3\rangle\cancelto{0}{[3|)|3]}\right) + \eta_{3A}\eta_{2A}\left(-|1\rangle\cancelto{0}{[1|)|1]}\right) = 0$$

4.18) Show that the supermomentum is conserved under the supershift

$|\hat{1}] = |1] + z|2]$, $|\hat{2}\rangle = |2\rangle - z|1\rangle$, $\hat{\eta}_{1A} = \eta_{1A} + z\eta_{2A}$

so that $\delta^{(8)}(\hat{Q})$ is invariant.

$$\sum_{i=1}^{n} |\hat{i}\rangle \hat{\eta}_{iA} = |1\rangle\hat{\eta}_{1A} + |\hat{2}\rangle\eta_{2A} + \sum_{i=3}^{n} |i\rangle\eta_{iA}$$

$$|1\rangle\hat{\eta}_{1A} + |\hat{2}\rangle\eta_{2A} = |1\rangle\eta_{1A} + z|1\rangle\eta_{2A} + |2\rangle\eta_{2A} - z|1\rangle\eta_{2A} = |1\rangle\eta_{1A} + |2\rangle\eta_{2A}$$

$$\Rightarrow \sum_{i=1}^{n} |\hat{i}\rangle \hat{\eta}_{iA} = \sum_{i=1}^{n} |i\rangle \eta_{iA} = 0$$

4.19) Show that $\langle i|K\cdot K|j\rangle = -K^2\langle ij\rangle$ for any momentum K (not necessarily null) & any spinors $|i\rangle, |j\rangle$.

$$\langle i|K\cdot K|j\rangle = \langle i|_{\dot a} K^{\dot a b} K_{b\dot c}|j\rangle^{\dot c} = \langle i|_{\dot a} K_\mu K_\nu (\bar\sigma^\mu)^{\dot a b}(\sigma^\nu)_{b\dot c}|j\rangle^{\dot c}$$

$$= \frac{1}{2}K_\mu K_\nu \left((\bar\sigma^\mu)^{\dot a b}(\sigma^\nu)_{b\dot c} + (\bar\sigma^\nu)^{\dot a b}(\sigma^\mu)_{b\dot c}\right)\langle i|_{\dot a}|j\rangle^{\dot c}$$

$$= \frac{1}{2}K_\mu K_\nu\left(-2\eta^{\mu\nu}\delta^{\dot a}{}_{\dot c}\right)\langle i|_{\dot a}|j\rangle^{\dot c} = -K^2\langle ij\rangle$$

uh yeah...
I sure hope it does

4.20) Write down the NMHV superamplitude formula that results from a $[2,3\rangle$ supershift. Then project out the gluon amplitude $A_6[+-+-+-]$. Can you match your result to the 3-term expression in Ex. 3.9?

#blaze it

I am a bee!

I do not know physics
all I know is bzz

$$\mathcal{A}_6^{\text{NMHV}} = \mathcal{A}_6^{\text{MHV}}\left(R_{246} + R_{241} + R_{251}\right)$$

$$A_6[+-+-+-] = \left(\frac{\partial}{\partial\eta_2}\right)^4\left(\frac{\partial}{\partial\eta_4}\right)^4\left(\frac{\partial}{\partial\eta_6}\right)^4\left(\mathcal{A}_6^{\text{MHV}}\left(R_{246}+R_{241}+R_{251}\right)\right)$$

$$= \int d^4\eta_2\, d^4\eta_4\, d^4\eta_6 \frac{\delta^{(8)}(\tilde{Q})}{\langle 12\rangle\cdots\langle 61\rangle}\left(R_{246}+R_{241}+R_{251}\right)$$

$$R_{246}: \int d^{12}\eta_{2,4,6} \frac{\delta^{(8)}\left(|2\rangle\eta_2 + |4\rangle\eta_4 + |6\rangle\eta_6 + \sum|i\rangle\eta_i\right)\langle 34\rangle\langle 56\rangle\delta^{(4)}(\Xi_{246})}{\langle 12\rangle\cdots\langle 61\rangle\, y_{46}^2\langle 2|y_{24}\cdot y_{26}|6\rangle\langle 2|y_{24}\cdot y_{46}|5\rangle\langle 2|y_{26}\cdot y_{64}|4\rangle\langle 2|y_{26}\cdot y_{64}|3\rangle}$$

$=$

$$\underline{\delta^{(8)}(\tilde{Q})}$$

$\langle 24\rangle\eta_{21}\eta_{41}$, $\langle 26\rangle\eta_{22}\eta_{62}$

$\eta_{23}\,\eta_{i3}\,\langle 2i\rangle$

Next steps: Chapter 5, 9, 10, 12, 13