Entropy and Temperature in de Sitter Space

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Abstract

We discuss several aspects of entropy and temperature in de Sitter space (dS). It is well known that, due to its cosmological horizon, dS possesses a natural temperature of $1/2\pi$, which should correspond to Hawking's gravitational entropy, given by $S = \text{Area}/4$. We engage with two problems of interest: 1) whether the gravitational entropy and temperature of dS can be accounted for via quantum entanglement; and 2) how the notion of temperature is affected by the nonstatic nature of dS. After reviewing the original derivation of Hawking temperature of dS, we compute a quantity known as entanglement entropy. Such an entropy arises from quantum entanglement between causally separated regions of the manifold, and the physical interpretation is simple: it enumerates the possible quantum states of the part of the universe beyond the horizon. We show that, while the Hawking temperature may be derived from entanglement entropy at fixed time slices, it is incompatible with the notion of time evolution in the hyperbolic slicing of dS. Our result for the entropy also suggests that entanglement entropy cannot fully account for the gravitational entropy of dS, as it does not scale with horizon area.

This paper represents my work in accordance with University regulations.
/s/ Loki Lin

Contents

1 Introduction

de Sitter space (dS) is a well-known cosmological spacetime that describes an expanding universe, similar to our own. One important feature of dS is an observer-dependent cosmological horizon, across which the observer cannot send information. Such a horizon arises from the rapid expansion of the spatial dimensions of dS.

There is some similarity between dS and Rindler space, which describes flat space from the perspective of accelerating observers. Rindler space also possesses an observer-dependent horizon that divides the manifold into causally separated regions. In 1976, Unruh [1] used this quality of Rindler space to show that an observer moving along a constant-acceleration trajectory in the Minkowski vacuum observes thermal radiation at a finite temperature of $a/2\pi$ where a is the proper acceleration of the observer. This phenomenon, known as the Unruh effect, is analogous to the fact that black holes emit thermal radiation. In fact, it has been shown [2] that near the event horizon of black holes, the Hawking temperature may be recovered by approximating a as the surface gravity of the black hole.

Gibbons and Hawking [3] showed in 1977 that cosmological spacetimes such as dS possess a natural temperature as well. The associated thermal radiation bears more similarity to that of Rindler rather than Schwarzchild space, because an observer in dS sees radiation coming from all directions, while an observer in Schwarzchild sees radiation coming from a fixed location, the black hole.

Temperature is closely tied to entropy, and Hawking's famous formula for horizon entropy $S = \text{Area}/4$, originally derived in the context of black holes, also applies to the cosmological horizon of dS [4]. In any situation involving a causal horizon, one may interpret the associated entropy as the number of states that the unobservable part of the universe may be in. More concisely, it arises from lack of information about the part of the universe beyond the horizon.

However, it is not always clear what states the "gravitational" entropy of the horizon actually describes. Often one introduces the more concrete notion of "geometric" or entanglement entropy to elucidate this concept. Entanglement entropy occurs when an observer only has partial knowledge about the quantum state of a system. It is natural to see how this arises in spacetimes with causal horizons. An observer in one region of the spacetime is only able to measure the part of a quantum system that is within his or her causal patch, and could not possibly have knowledge of the quantum state beyond a causal horizon.

Generally speaking, one may calculate this entropy explicitly by considering a quantum field in its vacuum state and tracing out the degrees of freedom in the unobservable part of the spacetime. Through this process, one obtains a density matrix describing the observable part of the field, from which the entropy is then computed. One question of interest is: can the density matrix be associated with a temperature? In Rindler space the answer is yes; one simply writes down the "entanglement Hamiltonian" which is the sum of the Hamiltonians for each mode, and one finds that this Hamiltonian defines a time evolution consistent with the choice of time coordinate for the Rindler wedge. The natural temperature of Rindler is then defined by matching Boltzmann factors to elements of the density matrix. In dS one can form the entanglement Hamiltonian, but it is not straightforward to determine whether it is compatible with time evolution. Thus, it is not so clear whether the associated temperature is actually physical.

Another interesting subtlety in the comparison between Rindler and dS is that the perceived particle creation in the Unruh effect is only due to the presence of a causal horizon, and not due to time evolution, because Rindler space is static. However, in nonstatic spacetimes such as dS, time evolution can source particle creation as well. It may be the case that thermal radiation in dS is partially caused by time evolution and not simply the cosmological horizon.

This paper is structured as follows: Section 2 develops the background necessary to study entanglement entropy in dS and also provides a derivation of the Unruh effect for the purpose of comparison with dS; Section 3 reviews the original derivation of the Hawking temperature in dS; Section 4 computes the entanglement entropy; and Section 5 addresses the question of whether the Hawking temperature can be viewed as a result of entanglement. We will offer some new perspectives on the entanglement entropy calculation and the associated entanglement Hamiltonian.

We work in natural units with $\hbar = c = G = k_{\mathrm{B}} = 1$.

2 Relevant background

We begin by reviewing some preliminaries regarding de Sitter space, quantum field theory, and statistical mechanics.

2.1 de Sitter space

This section draws from [4].

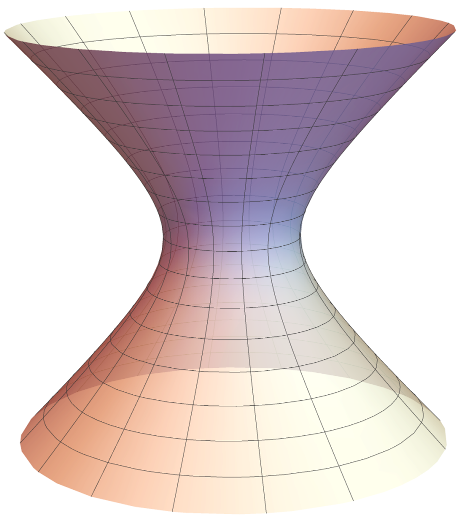

Figure 1: dS visualized as a hyperboloid of one sheet. Each circular slice represents a scaled copy of S^{n-1}.

Globally, n-dimensional de Sitter space is described as a hyperboloid embedded in $(n+1)$-dimensional Minkowski space. As we may recall, the $(n+1)$-dimensional Minkowski metric is given by

$$\mathrm{d}s^2 = -\mathrm{d}X_0^2 + \sum_{i=1}^{n} \mathrm{d}X_i^2 \tag{2.1}$$

The de Sitter manifold is defined by the relation

$$-X_0^2 + \sum_{i=1}^{n} X_i^2 = \ell^2 \tag{2.2}$$

The parameter ℓ is called the de Sitter radius. In everything that follows, we will set $\ell = 1$. We may parametrize dS by the following:

$$\begin{aligned} X_0 &= \sinh\tau \\ X_i &= \omega_i \cosh\tau,\ i = 1, \ldots, n \end{aligned} \tag{2.3}$$

where ω_i is the usual parametrization of the unit (n–1)-sphere. This induces the metric on dS:

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \cosh^2\tau\, \mathrm{d}\Omega_{n-1}^2 \tag{2.4}$$

Geometrically, this describes an (n–1)-sphere which has a minimum size at $\tau = 0$ and grows to infinite size as $\tau \to \pm\infty$.

The causal structure of dS is more apparent in static coordinates, which are analogous to Schwarzchild coordinates for black holes. They are obtained by the coordinate transformation

$$\begin{aligned} X_0 &= \sqrt{1-\bar{r}^2}\sinh\bar{t} \\ X_i &= \bar{r}\omega_i,\ i = 1, \ldots, n-1 \\ X_n &= \sqrt{1-\bar{r}^2\cosh\bar{t}} \end{aligned} \tag{2.5}$$

The metric is then

$$\mathrm{d}s^2 = -(1-\bar{r}^2)\mathrm{d}\bar{t}^2 + \frac{\mathrm{d}\bar{r}^2}{1-\bar{r}^2} + \bar{r}^2\mathrm{d}\Omega_{n-2}^2 \tag{2.6}$$

We may observe that the metric is singular at $\bar{r} = 1$. This is known as the de Sitter horizon. If we consider the trajectory of a photon moving radially outwards from $\bar{r} = 0$, we have

$$\frac{\mathrm{d}\bar{r}}{\mathrm{d}\bar{t}} = 1 - \bar{r}^2 \to 0 \text{ as } \bar{r} \to 1 \tag{2.7}$$

This demonstrates that an observer in this coordinate patch cannot send a signal beyond the $\bar{r} = 1$ horizon.

It is also important to emphasize that this coordinate system does not cover all of dS – it only covers the southern diamond (S), i.e. the intersection of the causal past and causal future of an observer at the south pole. In other words, this is the region of spacetime that is completely causally accessible to the observer in that he or she may send and receive signals from anywhere in the diamond. For this reason it is known as the causal or static patch.

In these coordinates, the metric is independent of $\bar{t}$, so $\partial_{\bar{t}}$ is a Killing vector field. In other words, the transformation it generates, time evolution, is a symmetry of the manifold. Figure 2 depicts the flow of ∂_t extended to the other regions. We note that $\partial_{\bar{t}}$ is timelike in the southern diamond, but globally it is not. Indeed, dS does not possess a globally timelike Killing vector, a fact which will be important when we later consider quantum fields in dS.

For our purposes it will be useful to define another set of coordinates on dS, known

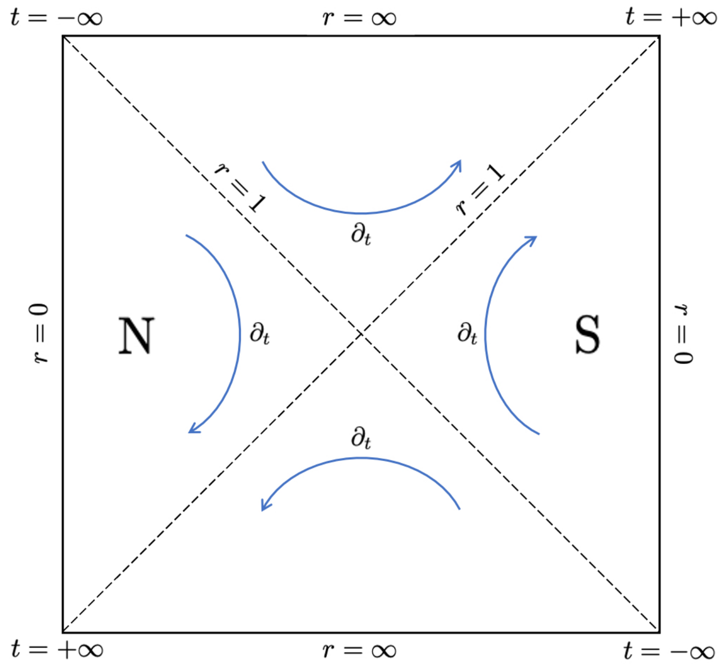

Figure 2: Conformal diagram of dS. The static patch Killing vector ∂_t is extended to the entire spacetime. In S it is timelike and future-oriented; in N it is timelike and past-oriented; elsewhere it is spacelike, and it vanishes at $r = 1$.

as the hyperbolic slicing. The parametrization is

$$\begin{aligned} X_0 &= \sinh t \cosh r \\ X_i &= \omega_i \sinh t \sinh r, \;\; i = 1, \ldots, n{-}1 \\ X_n &= \cosh t \end{aligned} \tag{2.8}$$

In these coordinates the metric is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \sinh^2 t(\mathrm{d}r^2 + \sinh^2 r \, \mathrm{d}\Omega^2_{n-2}) \tag{2.9}$$

As the name suggests, each slice of constant time is an $(n–2)$-dimensional hyperboloid, so geometrically this is an expanding hyperboloid in time which reaches infinite size as $t \to \pm\infty$. This is illustrated in Figure 3.

Written this way, the metric is clearly not static. However, if we consider late times $(t \to \infty)$, for small r the metric becomes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{e^{2t}}{2}(\mathrm{d}r^2 + r^2 \, \mathrm{d}\Omega^2_{n-2}) \tag{2.10}$$

We can make this metric look effectively static by introducing a change of radial variable:

$$\tilde{r} = \frac{e^t}{\sqrt{2}} r \tag{2.11}$$

In these new coordinates, the metric is approximately

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\tilde{r}^2 + \tilde{r}^2 \, \mathrm{d}\Omega^2_{n-2} \tag{2.12}$$

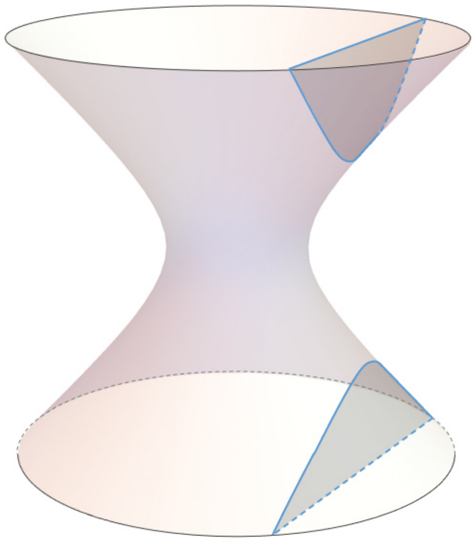

Figure 3: A hyperbolic time slice of dS. The two shaded regions each represent one sheet of H^{n-2} with metric (2.9).

Thus, ∂_t is a timelike Killing vector at late times. We expect that dS should behave like a static spacetime in this limit, which will motivate some analysis in Section 5.2.

A last set of coordinates which we will also find useful is Kruskal coordinates, related to static coordinates by the following:

$$\begin{aligned} \ln U &= t - \frac{1}{2}\ln\frac{1-r}{1+r} \\ -\ln(-V) &= t + \frac{1}{2}\ln\frac{1-r}{1+r} \end{aligned} \tag{2.13}$$

Note that the t, r that appear here are the static coordinates from which we are dropping the overlines, and not the hyperbolic coordinates. The metric takes the form

$$\frac{1}{(1-UV)^2}\big(-4\mathrm{d}U\mathrm{d}V + (1+UV)^2\mathrm{d}\Omega_{n-2}^2\big) \tag{2.14}$$

If we allow t to take on complex values, then U, V are periodic in imaginary time. To see this, consider the transformation $t \to t + 2\pi i n$ for some $n \in \mathbb{Z}$. Since $e^{2\pi i n} = 1$, the coordinates U, V are unchanged.

Notably, Rindler space possesses a similar type of periodicity, from which one may anticipate the Unruh effect. In Section 3, we will observe the implications of the periodicity of dS.

2.2 Quantum field theory in curved space

This section draws from [5] and [6].

We will use the method of second quantization to formulate quantum field theory in curved spacetime. The general procedure is as follows. We first derive a classical field equation, known as the Klein-Gordon equation. Then we elevate the field to a quantum operator by imposing canonical commutation relations. We may then choose a ground state and use it to define the Fock basis for the space of quantum states of the field.

We proceed with a basic derivation from classical field theory: the equation of motion for a scalar field ϕ in a curved space with metric $g_{\mu\nu}$. The action is given by

$$S = \int \mathrm{d}^n x \sqrt{-g}\mathcal{L}(\phi, \nabla_\mu \phi) \tag{2.15}$$

where the Lagrangian is

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \cdot \nabla_\nu \phi - V(\phi) \tag{2.16}$$

Setting $\delta S = 0$ gives the equation of motion

$$-\frac{\partial V}{\partial \phi} + \Box \phi = 0 \tag{2.17}$$

For a massive field the potential in curved space is $V = \frac{1}{2}(m^2 + \xi R)\phi^2$, so we arrive at the Klein-Gordon equation for massive scalar fields:

$$\Box \phi - (m^2 + \xi R)\phi = 0 \tag{2.18}$$

where ξ is a dimensionless parameter which sets the strength of the coupling of the field to the curvature. Actually, without loss of generality we can just set $\xi = 0$, because we can always absorb the coupling term into the mass, i.e. if $\xi \neq 0$ we instead use $\tilde{m} = \sqrt{m^2 + \xi R}$. Thus the Klein-Gordon equation is simply

$$\Box \phi - m^2 \phi = 0 \tag{2.19}$$

Now we would like to quantize the field. First we define a Hilbert space on the set of solutions via the inner product

$$\langle \psi_1 | \psi_2 \rangle = -i \int_\Sigma \mathrm{d}^{n-1} x \sqrt{-\tilde{g}}\, v^\mu (\psi_1 \nabla_\mu \psi_2^* - \psi_2 \nabla_\mu \psi_1^*) \tag{2.20}$$

where the surface of integration, Σ, is an arbitrary spacelike hypersurface, and $v^\mu, \tilde{g}_{\mu\nu}$ are the corresponding unit normal vector and induced metric. One can check that this inner product is independent of Σ, so it is indeed well defined.

As with any Hilbert space, there exists an orthonormal basis $\{u_i\}$ for our solution space. The basis may not be countable, but for our derivation we will assume it is; the uncountable case is easy to treat analogously. We expand ϕ in this basis:

$$\phi = \sum_i (a_i u_i + a_i^\dagger u_i^*) \tag{2.21}$$

We now impose the canonical commutation relations

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i, a_j^\dagger] = \delta_{ij} \tag{2.22}$$

Thus, the coefficients $a_i, a_i^\dagger$ become raising and lowering operators. The ground state of this basis is defined by the condition $a_i|0\rangle_u = 0 \, \forall \, i$. Subsequent excitation states $|\{n_i\}\rangle_u$ are given by acting on the ground state with the appropriate raising operators:

$$|\{n_i\}\rangle_u = \Big(\prod_i \frac{1}{\sqrt{n_i!}} (a_i^\dagger)^{n_i} \Big) |0\rangle_u \tag{2.23}$$

At this point, it is important to keep in mind that the choice of basis $\{u_i\}$ is not unique. In flat space, we are accustomed to choosing modes such that u_i are positive frequency and u_i^* are negative frequency with respect to the time coordinate, which is possible because ∂_t is a Killing vector. In particular, this implies that we can define a Hamiltonian that generates time translations; in other words there exists $\mathcal{H}$ such that $\frac{\mathrm{d}A}{\mathrm{d}t} = i[\mathcal{H}, A] + \left(\frac{\partial A}{\partial t}\right)_{\mathcal{H}}$. However, in curved space this is generally not true, as ∂_t is not necessarily a Killing vector, so there is no "natural" vacuum state in that sense.

Another perspective is to observe that if ∂_t is not a Killing vector, i.e. if $g_{\mu\nu}$ has t-dependence, then energy is not conserved. Thus, particles may be created as time evolves. For de Sitter space in particular, we recall that the spacelike components are expanding in time. The implication for a quantum field in dS is that new modes are constantly being created as space expands. This is an intuitive explanation for why energy is not conserved in dS, as the vacuum energy increases when modes are added.

We may introduce an observer to make the detection of particles more concrete. Let $\mathcal{O}$ be such an observer with trajectory $x^\mu(\tau)$, where τ is its proper time. Suppose that there exist a complete set of modes v_k, v_k^* corresponding to operators $b_k, b_k^\dagger$ such that

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\nabla_\mu v_j(x^\mu) = -i\omega_k v_k(x^\mu), \quad \phi = \sum_k (b_k v_k + b_k^\dagger v_k^*) \tag{2.24}$$

Then we say that $b_k^\dagger b_k$ is the number operator for particles in mode k detected by this observer, i.e. if the field is in a state $|\psi\rangle$, then $\mathcal{O}$ will detect $\langle\psi|b_k^\dagger b_k|\psi\rangle$ in the kth mode. As before, we can define a vacuum state $|0\rangle_v$ such that $b_k|0\rangle_v = 0 \;\forall\, k$. Via the raising operators $b_k^\dagger$, we obtain a corresponding basis of states $|\{m_i\}\rangle_v$.

Let us now relate the b_k operators to the a_j operators from before. First we relate the u_j modes to the v_k modes:

$$v_k = \sum_j (\alpha_{kj} u_j + \beta_{kj} u_j^*) \tag{2.25}$$

where $\alpha_{kj} = \langle v_k|u_j\rangle$, $\beta_{kj} = -\langle v_k|u_j^*\rangle$. Plugging this into the expansion of ϕ, we find that the ladder operators are related by

$$b_k = \sum_j (\alpha_{jk}^* a_j - \beta_{jk}^* a_j^\dagger) \tag{2.26}$$

The coefficients α_{kj}, β_{kj} are known as Bogoliubov coefficients, and the process of transforming from the u_j basis to the v_k basis is known as a Bogoliubov transformation.

Now suppose the field is in the $|0\rangle_u$ state, the vacuum state of the a_j operators. We find that the observer $\mathcal{O}$ actually detects a nonzero particle number in the kth mode:

$$\begin{aligned}\langle 0|_u b_k^\dagger b_k|0\rangle_u &= \langle 0|_u \sum_{i,j} (\alpha_{ki} a_i^\dagger - \beta_{ki} a_i)(\alpha_{kj}^* a_j - \beta_{kj}^* a_j^\dagger)|0\rangle_u \\ &= \sum_{i,j} \beta_{ki}\beta_{kj}^* \langle 0|_u (a_j^\dagger a_i + \delta_{ij})|0\rangle_u = \sum_j |\beta_{ij}|^2\end{aligned} \tag{2.27}$$

One example of this occurs in the Unruh effect, in which an accelerating observer sees excitations in the Minkowski vacuum.

An important subtlety, as alluded to in the Introduction, is that there is a distinction between particle creation due to a change in time coordinate (e.g. coordinate time vs. proper time of some trajectory), and particle creation due to time evolution in a nonstatic coordinatization. Only the former can occur in a static spacetime such as Rindler space, but both may occur in nonstatic slicings of dS. We will see that this makes it much more difficult to reconcile Hawking temperature and entanglement entropy in dS.

2.3 Statistical and thermodynamic entropy

There are a number of different formulations of entropy which will each be useful to us in different scenarios. At the most basic level, entropy is a measure of the number of possible states of a system at a certain energy. More precisely, given the multiplicity as a function of energy, $\Omega(\mathcal{U})$, the entropy is

$$S(\mathcal{U}) = \ln \Omega(\mathcal{U}) \tag{2.28}$$

This formulation is known as the microcanonical ensemble, and S is the Boltzmann entropy. Temperature is then defined by

$$\frac{1}{T} = \frac{\partial S}{\partial \mathcal{U}} \tag{2.29}$$

We may also formulate entropy in the canonical ensemble via the partition function $Z = \sum_s e^{-\mathcal{U}_s/T}$, where the sum is over all possible states s. The Gibbs entropy is given by

$$S = -\sum_s \mathcal{P}_s \ln \mathcal{P}_s, \quad \mathcal{P}_s = \frac{e^{-\mathcal{U}_s/T}}{Z} \tag{2.30}$$

It is easy to generalize this to a quantum system. The quantum analogue of the classical probabilities $\mathcal{P}_s$ is the density matrix ρ. The von Neumann entropy is the generalization of the Gibbs entropy, and it is given by

$$S = -\mathrm{Tr}(\rho \ln \rho) \tag{2.31}$$

This is the formula we will use to compute entanglement entropy. Physically, entanglement entropy arises from incomplete knowledge of the quantum state of the system. In other words, it is a measure of the remaining quantum degrees of freedom.

2.4 Unruh effect

The Unruh effect in 2 dimensions is the simplest example of entropy and temperature arising from the disagreement of different observers' vacuum states. We will run through a quick schematic derivation following [5], mainly so that we can later compare it to the situation in dS.

The Minkowski metric is $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2$. We start by defining the Minkowski modes of the field operator:

$$\phi = \int \mathrm{d}k (a_k u_k + a_k^\dagger u_k^*) \tag{2.32}$$

The u_k modes are simply

$$u_k = \frac{1}{2\sqrt{\pi \omega_k}} e^{i(kx - \omega_k t)} \tag{2.33}$$

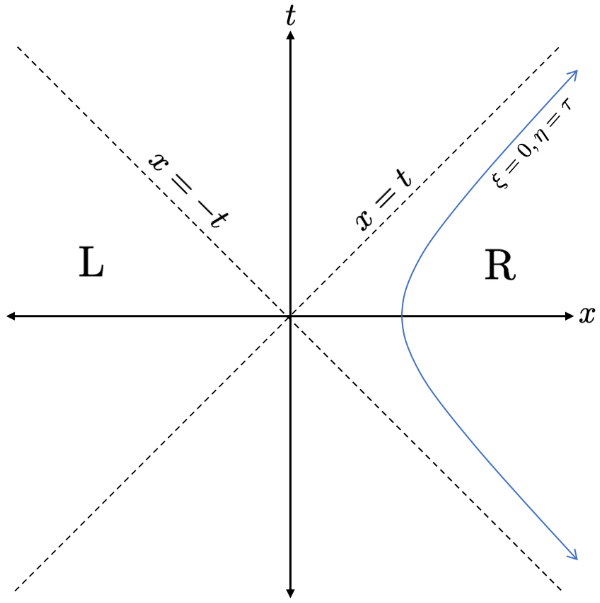

Figure 4: Rindler coordinates on Minkowski space. The wedges L and R have causal horizons at $x = \pm t$. The constant acceleration observer follows a trajectory of constant ξ. In this case, we have chosen $\xi = 0$.

where $\omega_k = \sqrt{k^2 + m^2}$. These are just the usual delta-function-normalized plane wave solutions of the Klein-Gordon equation in Minkowski space. u_k is positive frequency and u_k^* is negative frequency; this separation can be made because ∂_t is a Killing vector.

We now make the switch to Rindler coordinates, related to the Minkowski coordinates by

$$\begin{aligned} t &= \begin{cases} \frac{1}{a} e^{a\xi} \sinh(a\eta) & x > |t| \\ -\frac{1}{a} e^{a\xi} \sinh(a\eta) & x < |t| \end{cases} \\ x &= \begin{cases} \frac{1}{a} e^{a\xi} \cosh(a\eta) & x > |t| \\ -\frac{1}{a} e^{a\xi} \cosh(a\eta) & x < |t| \end{cases} \end{aligned} \tag{2.34}$$

These two coordinate patches cover causally separated regions of the spacetime known as Rindler wedges, as displayed in Figure 4. We call the $x > |t|$ region the right Rindler wedge (R) and the $x < |t|$ region the left Rindler wedge (L). In each region the metric is $\mathrm{d}s^2 = e^{2a\xi}(-\mathrm{d}\eta^2 + \mathrm{d}\xi^2)$ and $\partial_{\pm\eta}$ is a timelike Killing vector; in R ∂_η is future-oriented, and in L $\partial_{-\eta}$ is future-oriented. The parameter a is the acceleration of a Rindler observer on a $\eta = \tau, \xi = 0$ trajectory as measured by an inertial Minkowski observer.

We can take mode functions that solve the Klein-Gordon equation in one wedge and are 0 on the other. These functions are given by

$$\begin{aligned} g_k^R &= \frac{1}{2\sqrt{\pi\omega_k}} \begin{cases} e^{i(k\xi - \omega_k \eta)} & \text{in R} \\ 0 & \text{in L} \end{cases} \\ g_k^L &= \frac{1}{2\sqrt{\pi\omega_k}} \begin{cases} 0 & \text{in R} \\ e^{i(k\xi + \omega_k \eta)} & \text{in L} \end{cases} \end{aligned} \tag{2.35}$$

These are positive frequency with respect to ∂_η in R and $\partial_{-\eta}$ in L. Let $b_k^{R\dagger}, b_k^R, b_k^{L\dagger}, b_k^L$ be the corresponding creation and annihilation operators for these modes.

To make the calculation easier, we introduce a new set of modes h_k which are the analytic extension of the g_k modes:

$$\begin{aligned} h_k^R &= \frac{1}{\sqrt{2\sinh(\pi\omega/a)}}\left(e^{\pi\omega/2a} g_k^R + e^{-\pi\omega/2a} g_{-k}^{L*}\right) \\ h_k^L &= \frac{1}{\sqrt{2\sinh(\pi\omega/a)}}\left(e^{\pi\omega/2a} g_k^L + e^{-\pi\omega/2a} g_{-k}^{R*}\right) \end{aligned} \tag{2.36}$$

where we have dropped the k subscripts on ω for simplicity. Let $c_k^{R\dagger}, c_k^R, c_k^{L\dagger}, c_k^L$ be the corresponding creation and annihilation operators. These are related to the b_k operators by

$$\begin{aligned} b_k^R &= \frac{1}{\sqrt{2\sinh(\pi\omega/a)}}\left(e^{\pi\omega/2a} c_k^R + e^{-\pi\omega/2a} c_{-k}^{L\dagger}\right) \\ b_k^L &= \frac{1}{\sqrt{2\sinh(\pi\omega/a)}}\left(e^{\pi\omega/2a} c_k^L + e^{-\pi\omega/2a} c_{-k}^{R\dagger}\right) \end{aligned} \tag{2.37}$$

Conveniently, the c_k^R, c_k^L operators annihilate the Minkowski vacuum, i.e. $|0_a\rangle = |0_c\rangle$. This makes it easy to calculate the expectation value of the R region number operator $b_k^{R\dagger} b_k^R$ in the Minkowski vacuum state:

$$\langle 0_a | b_k^{R\dagger} b_k^R | 0_a \rangle = \frac{1}{e^{2\pi\omega/a} - 1} \langle 1_{c,-k} | 1_{c,-k} \rangle = \frac{1}{e^{2\pi\omega/a} - 1} \delta(0) \tag{2.38}$$

This matches the Bose-Einstein distribution with temperature $T = a/2\pi$.

Alternatively, we could have followed the method suggested in the Introduction by finding $|0\rangle_a$ in terms of $|n_R\rangle_c \otimes |n_L\rangle_c$, forming the reduced density matrix for one of the regions, and interpreting the elements as Boltzmann factors, which would imply the Unruh temperature. We will take that approach in Section 4 for the dS calculation, but for the Rindler case it is unnecessarily complicated.

3 Hawking temperature of dS

This section uses some ideas from [6] and [3].

Again we consider a massive scalar field ϕ on dS. We recall the Kruskal coordinates from Section 2.1, related to the static coordinates by

$$\begin{aligned} \ln U &= t - \frac{1}{2}\ln\frac{1-r}{1+r} \\ -\ln(-V) &= t + \frac{1}{2}\ln\frac{1-r}{1+r} \end{aligned} \tag{3.1}$$

Before, we noted that these coordinates are periodic in imaginary static time with period 2π. As we discussed, the static time coordinate gives rise to a future-directed timelike Killing vector ∂_t, but this coordinate was originally only defined in the southern causal diamond. In Kruskal coordinates ∂_t is still a Killing vector, but it is only future-directed and timelike in the southern diamond. Thus, in this region we have a well-defined notion positive and negative frequency modes $u_j, u_j^\dagger$. Another implication is that there exists a well-defined Hamiltonian $\mathcal{H}$ that describes the evolution of ϕ in t, so we can associate an energy to each mode.

As usual, let us expand the field operator in these modes. We have

$$\phi = \sum_j (a_j u_j + a_j^\dagger u_j^*) \tag{3.2}$$

Because these modes separate into positive and negative frequency with respect to time, $\mathcal{H} = \sum_j \mathcal{H}_j = \sum_j a_j^\dagger a_j$ is a well-defined Hamiltonian. Let the states $|\{n_j\}\rangle$ denote the corresponding Fock basis. We may restrict to a j-subspace and drop the subscript in the following.

The state $|n\rangle$ has energy E_n defined by $\mathcal{H}|n\rangle = E_n|n\rangle$. Now, denoting $\beta = 1/T$, we can write the partition function of the canonical ensemble associated with this mode as

$$Z = \mathrm{Tr}(e^{-\beta\mathcal{H}}) = \sum_n e^{-\beta E_n} = \sum_n \langle n|e^{-\beta E_n}|n\rangle \tag{3.3}$$

We may note that, since the time evolution operator is $U(t) = e^{-i\mathcal{H}t}$, the expression $\langle n|e^{-\beta E_n}|n\rangle$ is the probability amplitude for the state $|n\rangle$ to return to itself after time evolution by $\Delta t = -i\Delta\tau = -i\beta$. The rotation of the real axis t into the imaginary axis $-i\tau$ in the complex plane is known as Wick rotation.

From this, we can write $Z = \sum_n \langle n|n(\Delta t)\rangle$. The density matrix of the ensemble state is then given by

$$\rho = \frac{1}{Z}e^{-\beta\mathcal{H}} = \frac{1}{Z}e^{-i\mathcal{H}\Delta t} \tag{3.4}$$

Given any operator $\mathcal{A}$, its expectation value in this state is

$$\langle\mathcal{A}\rangle = \mathrm{Tr}(\rho\mathcal{A}) = \frac{\sum_n \langle n|e^{-i\mathcal{H}\Delta t}\mathcal{A}|n\rangle}{\sum_m \langle m|e^{-i\mathcal{H}\Delta t}|m\rangle} \tag{3.5}$$

We can define the propagator of $\mathcal{A}$ as the (implicitly time-ordered) two point function

$$\begin{aligned} G_{\mathcal{A}}(\tau,\tau') &= \langle\mathcal{A}(\tau)\mathcal{A}(\tau')\rangle = \frac{1}{Z}\mathrm{Tr}\big(e^{-\beta\mathcal{H}}\mathcal{A}(\tau)\mathcal{A}(\tau')\big) \\ &= \frac{1}{Z}\mathrm{Tr}\big(e^{-(\tau+\beta)\mathcal{H}}\mathcal{A}(0)e^{\mathcal{H}\tau}\mathcal{A}(\tau')\big) = \frac{1}{Z}\mathrm{Tr}(\mathcal{A}(\tau+\beta)e^{-\beta\mathcal{H}}\mathcal{A}(\tau')\big) \\ &= \frac{1}{Z}\mathrm{Tr}\big(e^{-\beta\mathcal{H}}\mathcal{A}(\tau+\beta)\mathcal{A}(\tau')\big) \end{aligned} \tag{3.6}$$

where in the last step we have used the implicit time ordering and the invariance of trace under cyclic permutations. Thus we see that the two-point function, also known as the thermal Green's function, is periodic in imaginary time $\tau = it$ with period $\Delta\tau = \beta = 1/T$.

Now we would like to argue that the propagator of the quantum field ϕ in dS has a periodicity in imaginary time. We can construct this propagator $G_\phi(x^\mu, x'^\mu)$ via Kruskal coordinates U, V, but the details of the construction are unnecessary. What is key is that such a propagator can be analytically extended to imaginary time (except for points x^μ, x'^μ that have $g_{\mu\nu}(x'-x)^\nu(x'-x)^\mu = 0$, i.e. null-separated). Thus, because the Kruskal coordinates are periodic in t with period $2\pi i$, so must be $G_\phi(x^\mu, x'^\mu)$. This indicates that G_ϕ is a thermal Green's function. Matching the periods, we find that G_ϕ

describes a system at thermal equilibrium with temperature $T = 1/2\pi$. This temperature is independent of the mode quantum number j, so it is uniform across all modes.

Physically, this means that an observer in dS sees thermal radiation coming from the dS horizon, or in other words from all directions. We notice that this bears a degree of similarity with the Unruh effect in Rindler space. Indeed, we could have derived the Unruh temperature by noting that t and x as written in (2.34) are each periodic in imaginary η with period $2\pi i/a$, leading to a temperature of $a/2\pi$. In both cases, we may interpret the temperature as arising from some entropy which is the logarithm of the number of states of the unobservable universe and which satisfies $\frac{\partial S}{\partial \mathcal{U}} = T^{-1}$.

This notion of entropy is easy to formalize in the analogous case of black holes, where we may argue that $T^{-1} = \frac{\partial S}{\partial M}$, since the energy of the black hole is simply the mass. However, in the cosmological case, we are left with the question: What exactly are the states of the universe that this entropy enumerates?

4 Entanglement entropy in dS

We use the notion of entanglement entropy to provide a partial answer to the preceding question. Namely, we show that we may associate an entropy to the dS horizon by considering the vacuum state of a quantum field and tracing over the degrees of freedom in the causally separated region of dS. A brief schematic of the steps are as follows: 1) solve the Klein-Gordon equation in terms of modes; 2) find the ground state; and 3) form its density matrix and compute the entropy. This section will closely follow the calculation in [7].

4.1 Klein-Gordon modes

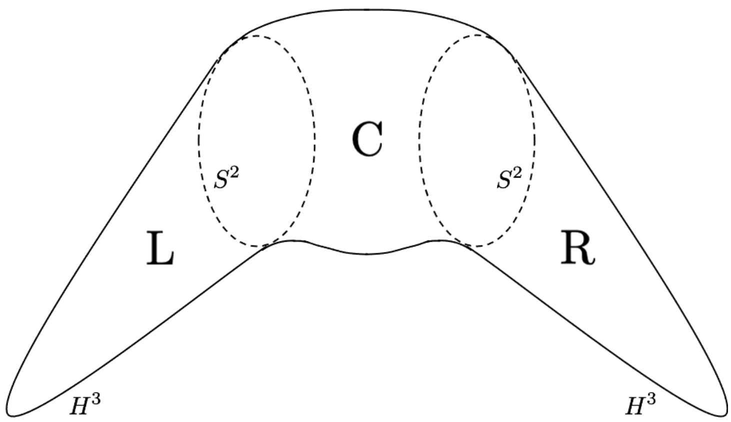

Figure 5: Division of a hyperbolic time slice of dS into regions L, R, and C using the metric (4.1). Each of the hyperboloids L and R are bounded by a spherical horizon, which keeps them causally separate. The central region C is part of the ambient space. The metric on the hyperboloids can be obtained by analytic continuation of the metric on C.

We may use the hyperbolic slicing to divide dS into three regions: left (L), right (R), and center (C), as shown in Figure 5. The metric on each of these regions is given by

$$\begin{aligned} \mathrm{d}s_q^2 &= -\mathrm{d}t_q^2 + \sinh^2 t_q(\mathrm{d}r_q^2 + \sinh^2 r_q\, \mathrm{d}\Omega_2^2),\ q \in \{\mathrm{L},\ \mathrm{R}\} \\ \mathrm{d}s_C^2 &= \mathrm{d}t_C^2 + \cos^2 t_C(-\mathrm{d}r_C^2 + \cosh^2 r_C\, \mathrm{d}\Omega_2^2) \end{aligned} \tag{4.1}$$

We may observe that L and R are causally separated, so the associated Hilbert spaces of quantum states are well-suited for our formulation of entanglement entropy. In these two regions we would like to solve the Klein-Gordon equation for a scalar field ϕ of mass m.

As usual, we expand the field operator in eigenmodes $u_{\sigma p \ell m}$ of the Klein-Gordon operator:

$$\phi = \int \mathrm{d}p \sum_{\sigma,\ell,m} (a_{\sigma p\ell m} u_{\sigma p\ell m} + a^\dagger_{\sigma p\ell m} u^*_{\sigma p\ell m}) \tag{4.2}$$

for suitable quantum numbers σ, p, ℓ, m. The Klein-Gordon equation then reads

$$\Big(\frac{1}{\sinh^3 t}\frac{\partial}{\partial t}\sinh^3 t\frac{\partial}{\partial t} - \frac{1}{\sinh^3 t}\nabla^2_{H^3} + \frac{9}{4} - \nu^2\Big)u(t,r,\Omega_2) = 0 \tag{4.3}$$

where $\nabla^2_{H^3}$ is the Laplacian on the 3-dimensional hyperboloid and $\nu = \sqrt{\frac{9}{4} - m^2}$ describes the coupling strength. The eigenfunctions are given by

$$u_{\sigma p\ell m} \sim \frac{1}{\sinh t}\chi_{\sigma p}(t) Y_{p\ell m}(r,\Omega_2) \tag{4.4}$$

where the Y_{plm} are the hyperbolic harmonics satisfying $\nabla_{H^3} Y_{p\ell m} = -(1+p^2)Y_{p\ell m}$, and $\sigma = \pm 1$. The $\chi_{\sigma p}$ have the following form in each region:

$$\chi_{\sigma p} = \frac{1}{2\sinh(\pi p)} \begin{cases} \frac{e^{\pi p} - i\sigma e^{-i\pi\nu}}{\Gamma(\nu+ip+1/2)} P^{(ip)}_{(\nu-1/2)}(\cosh t_R) \\ \qquad\qquad - \frac{e^{-\pi p} - i\sigma e^{-i\pi\nu}}{\Gamma(\nu-ip+1/2)} P^{(-ip)}_{(\nu-1/2)}(\cosh t_R) & \text{in R} \\ \sigma\Big[\frac{e^{\pi p} - i\sigma e^{-i\pi\nu}}{\Gamma(\nu+ip+1/2)} P^{(ip)}_{(\nu-1/2)}(\cosh t_L) \\ \qquad\qquad - \frac{e^{-\pi p} - i\sigma e^{-i\pi\nu}}{\Gamma(\nu-ip+1/2)} P^{(-ip)}_{(\nu-1/2)}(\cosh t_L)\Big] & \text{in L} \end{cases} \tag{4.5}$$

where Γ is the Euler-Gamma function and $P^{(n)}_{(m)}$ are the Legendre functions.

Our eventual task is to solve for the ground state of the $a_{\sigma p\ell m}$ operators, i.e. we would like to find $|\Psi\rangle$ such that $a_{\sigma p\ell m}|\Psi\rangle = 0\ \ \forall\, \sigma, p, \ell, m$. With this purpose in mind, we find it useful to define some basis functions for the combined R and L regions. We will use the Legendre functions, modified a bit:

$$\begin{aligned} P_R^{\pm p}(t_{R,L}) &= \begin{cases} P^{(\pm ip)}_{(\nu-1/2)}(\cosh t_R) & \text{in R} \\ 0 & \text{in L} \end{cases} \\ P_L^{\pm p}(t_{R,L}) &= \begin{cases} 0 & \text{in R} \\ P^{(\pm ip)}_{(\nu-1/2)}(\cosh t_L) & \text{in L} \end{cases} \end{aligned} \tag{4.6}$$

With this basis, $\chi_{\sigma p}$ has the form

$$\begin{aligned}\chi_{\sigma p} &= \frac{1}{N_p}\frac{\sigma}{2\sinh(\pi p)}\sum_{q\in\{R,L\}}\Big(\frac{e^{\pi p}-i\sigma e^{-i\pi\nu}}{\Gamma(\nu+ip+1/2)}P^p_q - \frac{e^{-\pi p}-i\sigma e^{-i\pi\nu}}{\Gamma(\nu-ip+1/2)}\overline{P}^p_q\Big) \\ &= \frac{1}{N_p}\sum_q(\alpha^\sigma_{qp}P^p_q + \beta^\sigma_{qp}\overline{P}^p_q)\end{aligned} \tag{4.7}$$

Since the radial and angular dependences are solved independently, we may now restrict to a p-subspace and drop the p, ℓ, m indices. In other words, we know that the density matrix for $|\Psi\rangle$ will be diagonal.

4.2 Ground state

We are now ready to find the ground state of the a^σ operators. For the ensuing computations it is helpful to define the quantities

$$M = \begin{pmatrix} \alpha^+_R & \alpha^+_L & \beta^+_R & \beta^+_L \\ \alpha^-_R & \alpha^-_L & \beta^-_R & \beta^-_L \\ \overline{\beta}^+_R & \overline{\beta}^+_L & \overline{\alpha}^+_R & \overline{\alpha}^+_L \\ \overline{\beta}^-_R & \overline{\beta}^-_L & \overline{\alpha}^-_R & \overline{\alpha}^-_L \end{pmatrix}, \quad \chi = \begin{pmatrix}\chi^{p+} \\ \chi^{p-} \\ \overline{\chi}^{p+} \\ \overline{\chi}^{p-}\end{pmatrix}, \quad P = \begin{pmatrix}P_R \\ P_L \\ \overline{P}_R \\ \overline{P}_L\end{pmatrix} \tag{4.8}$$

Then we have the matrix equation $\chi = \frac{1}{N}MP$. We may interpret the $\alpha^\sigma_q, \beta^\sigma_q$ as Bogoliubov coefficients for a transformation between the operators a_σ associated with basis functions χ_σ and some new operators b_q associated with basis modes P_q. ϕ is invariant under Bogoliubov transformations, so we can relate a_σ, b_q by

$$\phi = \int \mathrm{d}p \sum_{\ell,m}\sum_\sigma(a_\sigma\chi_\sigma + a^\dagger_\sigma\overline{\chi}_\sigma) = \int \mathrm{d}p\sum_{\ell,m}\frac{1}{N_p}\sum_q(b_qP_q + b^\dagger_q\overline{P}_q) \tag{4.9}$$

where the p, ℓ, m subscripts on each term are implied. We can compactify this further by requiring the integrands and each ℓ, m summand to be individually equal. This gives, in each p, ℓ, m subspace,

$$a_i\chi_i = \frac{1}{N}b_jP_j \tag{4.10}$$

Using the change-of-basis matrix M, we then have

$$a_iM_{ij} = b_j \tag{4.11}$$

So now we have a_i in terms of b_j:

$$a_i = b_j(M^{-1})_{ij} \tag{4.12}$$

We denote the elements of M^{-1} by the coefficients of the inverse Bogoliubov transformation $\gamma^\sigma_q, \varepsilon^\sigma_q$ such that

$$a^\sigma = \sum_q(\gamma^\sigma_q b_q + \overline{\varepsilon}^\sigma_q b^\dagger_q) \tag{4.13}$$

Recall that the ground state condition is $a^\sigma|\Psi\rangle = 0 \;\; \forall\,\sigma$. Let $|0_b\rangle = |R\rangle\otimes|L\rangle$ where $|R\rangle, |L\rangle$ are such that $b_R|R\rangle = b_L|L\rangle = 0$. Then $|0_b\rangle$ is the ground state of the b_q operators, i.e. $b_q|0_b\rangle = 0 \;\; \forall\, q$.

To find $|\Psi\rangle$ in terms of $|0_b\rangle$, we may try the ansatz

$$|\Psi\rangle = \exp\Big[\frac{1}{2}\sum_{q,s} m_{qs} b_q^\dagger b_s^\dagger\Big]|0_b\rangle \tag{4.14}$$

for some symmetric (not necessarily Hermitian) matrix m. We require $a^\sigma|\Psi\rangle = 0$ (dropping the σ subscripts for the time being):

$$\begin{aligned} a|\Psi\rangle &= \Big(\sum_k \gamma_k \frac{\partial}{\partial b_k^\dagger} + \bar{\varepsilon}_k b_k^\dagger\Big) \exp\Big[\frac{1}{2}\sum_{q,s} m_{qs} b_q^\dagger b_s^\dagger\Big]|0_b\rangle \\ &= \Big[\sum_k \gamma_k \frac{1}{2}\Big(\sum_s m_{ks} b_s^\dagger + \sum_q m_{qk} b_q^\dagger\Big) + \bar{\varepsilon}_k b_k^\dagger\Big]|\Psi\rangle \\ &= \Big(\sum_{k,q} \gamma_k m_{qk} b_q^\dagger + \sum_q \bar{\varepsilon}_q b_q^\dagger\Big)|\Psi\rangle = 0 \end{aligned} \tag{4.15}$$

Thus we must have

$$\sum_{k,q} \gamma_k m_{qk} b_q^\dagger + \sum_q \bar{\varepsilon}_q b_q^\dagger = 0 \tag{4.16}$$

which gives the condition

$$\gamma_k m_{qk} = -\bar{\varepsilon}_q \tag{4.17}$$

or $m_{qk} = -\bar{\varepsilon}_q^\sigma(\gamma^{-1})_k^\sigma$. In addition to the symmetric condition $m_{RL} = m_{LR}$, we also have $m_{RR} = m_{LL}$, because γ_L, ε_L both differ from γ_R, ε_R by a sign. Directly computing these coefficients, we find

$$\begin{aligned} m_{RR} &= -\frac{2e^{-\pi p}\cos(\pi\nu)}{\cosh\big(\pi(p - i\nu)\big)} \\ m_{RL} &= -\frac{2ie^{-\pi p}\sinh(\pi p)}{\cosh\big(\pi(p - i\nu)\big)} \end{aligned} \tag{4.18}$$

It is now useful to perform yet another Bogoliubov transformation. We would like our state to have the form $|\Psi\rangle = \exp\big[\lambda c_R^\dagger c_L^\dagger\big]|0_c\rangle$ for some operators c_R, c_L and states $|0_c\rangle = |R'\rangle \otimes |L'\rangle$ such that $c_R|R'\rangle = c_L|L'\rangle = 0$. We have the relations

$$\begin{aligned} c_R|\Psi\rangle &= \frac{\partial}{\partial c_R^\dagger}|\Psi\rangle = \lambda c_L^\dagger|\Psi\rangle \\ c_L|\Psi\rangle &= \frac{\partial}{\partial c_L^\dagger}|\Psi\rangle = \lambda c_R^\dagger|\Psi\rangle \end{aligned} \tag{4.19}$$

Let $c_R = wb_R + vb_R^\dagger,\ c_L = \overline{w}b_L + \overline{v}b_L^\dagger$. Denoting $\eta = m_{RR} = m_{LL}, \zeta = m_{RL} = m_{LR}$, we have

$$\begin{aligned} &(w\frac{\partial}{\partial b_R^\dagger} + vb_R^\dagger)|\Psi\rangle = ((v + w\eta)b_R^\dagger + w\zeta b_L^\dagger)|\Psi\rangle \\ &= \lambda(wb_L^\dagger + v\frac{\partial}{\partial b_L^\dagger})|\Psi\rangle = \lambda(v\zeta b_R^\dagger + (v\eta + w)b_L^\dagger)|\Psi\rangle \end{aligned} \tag{4.20}$$

This implies the system of equations

$$
\begin{aligned}
w\eta + (1-\lambda\zeta)v &= 0 \\
w(\zeta-\lambda) - \lambda\eta v &= 0
\end{aligned}
\tag{4.21}
$$

In order for this system to have nontrivial solutions for w, v, for each p we must have

$$
\begin{aligned}
\lambda &= \frac{1+\zeta^2-\eta^2 \pm \sqrt{(\eta^2-\zeta^2-1)^2-4\zeta^2}}{2\zeta} \\
&= \frac{\sqrt{2}i}{\sqrt{\cosh(2\pi p)+\cos(2\pi\nu)}+\sqrt{\cosh(2\pi p)+\cos(2\pi\nu)+2}}
\end{aligned}
\tag{4.22}
$$

We also impose $|w|^2 - |v|^2 = 1$ in order to preserve the canonical commutation relations for the c_q operators. Solving for w, v now gives

$$
\begin{aligned}
w &= i\frac{\lambda\zeta - 1}{\sqrt{\eta^2-(\lambda\zeta-1)^2}} \\
v &= i\frac{\eta}{\sqrt{\eta^2-(\lambda\zeta-1)^2}}
\end{aligned}
\tag{4.23}
$$

Let us pause here and summarize what we have done by making an analogy to the Minkowski/Rindler case. The state $|\Psi\rangle$ is the analogue of the Minkowski vacuum. We then transformed to another set of operators b_q which operate only on half of dS. $|\Psi\rangle$ in terms of these operators was a bit unwieldy, so we transformed again to the operators c_q, which also operate on half of dS, to get a slightly simpler form for $|\Psi\rangle$. These operators of the analogue of the creation and annihilation operators on each wedge of Rindler space. The most important takeaway at this point is that $|\Psi\rangle$ is emphatically not equal to the vacuum state of the c_q operators. This is how temperature and entropy arise in dS, analogous to the Unruh effect in Rindler space.

4.3 Density matrix and entropy

Our ground state has the form

$$
|\Psi\rangle = \exp\left[\lambda c_R^\dagger c_L^\dagger\right]|0_c\rangle \tag{4.24}
$$

The density matrix in the p, ℓ, m subspace is now easy to write down:

$$
\rho = \frac{1}{Z}|\Psi\rangle\langle\Psi| = \exp\left[\lambda c_R^\dagger c_L^\dagger\right]|0_c\rangle\langle 0_c|\exp\left[\overline{\lambda} c_R c_L\right] \tag{4.25}
$$

Now we trace over the R Hilbert space to get the reduced density matrix for the L region. This reduced density matrix describes the part of the system that we can measure, i.e. the L region, while accounting for the extra degrees of freedom in the R region that we have no way of accessing.

$$\begin{aligned}\rho_L = \mathrm{Tr}_R(\rho) &= \frac{1}{Z}\sum_{n=0}^{\infty}\langle R'_n|e^{\lambda c_R^\dagger c_L^\dagger}|R'_0\rangle|L'_0\rangle\langle L'_0|\langle R'_0|e^{\overline{\lambda}c_L c_R}|R'_n\rangle \\ &= \frac{1}{Z}\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}c_L^{\dagger\, n}|L'_0\rangle\langle L'_0|c_L^n\frac{\overline{\lambda}^n}{\sqrt{n!}} = \frac{1}{Z}\sum_n\frac{|\lambda|^{2n}}{n!}\sqrt{n!}|L'_n\rangle\langle L'_n|\sqrt{n!} \\ &= \frac{1}{Z}\sum_n|\lambda|^{2n}|L'_n\rangle\langle L'_n| = \frac{1}{Z}\sum_n|\lambda_p|^{2n}|L'_n\rangle_{p\ell m}\langle L'_n|_{p\ell m}\end{aligned} \tag{4.26}$$

Enforcing $\mathrm{Tr}(\rho_L) = 1$ we get the normalization

$$Z = \frac{1}{1-|\lambda_p|^2} \tag{4.27}$$

Thus for each p-subspace we have the entropy

$$S(p) = -\mathrm{Tr}\big(\rho_L(p)\ln\rho_L(p)\big) = -\ln(1-|\lambda_p|^2) - \frac{|\lambda_p|^2}{1-|\lambda_p|^2}\ln|\lambda_p|^2 \tag{4.28}$$

In order to get the total entropy, we must integrate over p:

$$S_{\mathrm{tot}} = V_{H^3}\int \mathrm{d}p\, D_{H^3}(p)S(p) = V_{H^3}\int \mathrm{d}p\,\frac{p^2}{2\pi^2}S(p) \tag{4.29}$$

where V_{H^3} is the volume of the hyperboloid and $D_{H^3}(p)$ is the density of states for radial functions on the hyperboloid. We would need to regularize V_{H^3} and establish a UV cutoff for p in order to get a sensible result.

We go back to our original question: can we associate a temperature to this entropy? If so, how "physical" is it? These questions are the focus of the next section.

5 Reconciling entropy and temperature

Our goal is to recover the dS temperature $T = 1/2\pi$. The brute force way would be to use an Unruh detector with coupling Hamiltonian $\mathcal{H}_{\mathrm{int}} = gM(\tau)\phi$, where τ is the proper time along the detector's trajectory [6]. To first order, the probability of the detector to have an excitation ΔE from the ground state would then be proportional to

$$P_{\Delta E} \sim \Big|\int \mathrm{d}\Delta\tau\; e^{i\Delta E\Delta\tau}G(\tau,\tau')\Big|^2 \tag{5.1}$$

where G is the Feynman propagator, given by

$$G(\tau,\tau') = \langle\Psi|\mathcal{T}\big(\phi(x^\mu(\tau))\phi(x^\mu(\tau'))\big)|\Psi\rangle \tag{5.2}$$

In the flat space case the propagator is quite simple to find explicitly. However, in the hyperbolic slicing of dS, the calculation is much more complicated and not very physically enlightening, so we look to other methods.

5.1 Entanglement Hamiltonian

We recall that, in order to have a temperature, we must be able to define the energy of a mode; thus, what we really need is a Hamiltonian. Following [7], we may define what is known as the entanglement Hamiltonian for each mode of the quantum state:

$$\mathcal{H}_p = -\frac{\ln|\lambda_p|^2}{2\pi} c^\dagger_{L_p} c_{L_p} \tag{5.3}$$

The Hamiltonian of the entire system is simply the integral of this over p.

Let $E_p = -\ln|\lambda_p|^2/2\pi$. Acting on the states $|L'_n\rangle_p$ gives the energies

$$\mathcal{H}_p|L'_n\rangle_p = nE_p|L'_n\rangle_p \tag{5.4}$$

Thus the energy of the p-th mode with occupancy n is $E_{p_n} = nE_p$.

Now we look at the form of the density matrix ρ_L (4.26) and note that it corresponds to a canonical ensemble with Boltzmann factors $e^{-\beta E_{p_n}} = |\lambda_p|^{2n}$. Solving for the temperature gives

$$T = 1/\beta = -\frac{E_p}{\ln|\lambda_p|^2} = \frac{1}{2\pi} \tag{5.5}$$

which is exactly what we wanted.

5.2 Time evolution

We have used the entanglement Hamiltonian to recover the de Sitter temperature by construction. However, the dynamical meaning of this Hamiltonian is not clear. Namely, does it define a sensible time evolution? To answer this question, we can compute the commutator $[\mathcal{H}_p, \phi]$. If $\mathcal{H}$ is the dynamical Hamiltonian of the system, we would expect $\frac{\mathrm{d}\phi}{\mathrm{d}t} = i[\mathcal{H}, \phi]$.

We proceed with a small modification: the entanglement Hamiltonian (5.3) only acts on region L; we want to extend it to region R so that we can study time evolution of the entire field. Hence, we redefine the Hamiltonian to be

$$\mathcal{H} = E_p(c^\dagger_{L_p} c_{L_p} + c^\dagger_{R_p} c_{R_p}) \tag{5.6}$$

One might be concerned that this redefinition would change the analysis of the previous section. But there is actually no issue: the R term simply shifts the $|L_n\rangle$ energies by an overall constant, and we only care about energy differences so we can throw it away.

Again it will be sufficient to just work in a $p\ell m$-subspace. Also, we will use the more convenient P_R, P_L modes, the reasons for which will become clear shortly. In this basis the field operator is given by

$$\phi = \frac{1}{N \sinh t} \sum_q (b_q P_q + b^\dagger_q P^*_q) \tag{5.7}$$

We now compute

$$
\begin{aligned}
i[\mathcal{H},\phi] &= -i\frac{\ln|\lambda|^2}{2\pi N\sinh t}\sum_q\Big([c_R^\dagger c_R + c_L^\dagger c_L, b_q]P_q + [c_R^\dagger c_R + c_L^\dagger c_L, b_q^\dagger]P_q^*\Big)\\
&= i\frac{\ln|\lambda|^2}{2\pi N\sinh t}\Big(b_R\big((|w|^2 - w\overline{v})P_R + (|v|^2-\overline{v}w)P_R^*\big)\\
&\qquad + b_R^\dagger\big((\overline{w}v - |w|^2)P_R + (v\overline{w} - |v|^2)P_R^*\big)\\
&\qquad + b_L\big((|w|^2 - \overline{w}v)P_L + (|v|^2 - v\overline{w})P_L^*\big)\\
&\qquad + b_L^\dagger\big((w\overline{v} - |w|^2)P_L + (\overline{v}w - |v|^2)P_L^*\big)\Big)
\end{aligned}
\tag{5.8}
$$

Thus we see that the commutator causes mixing of the mode functions.

On the other hand, the time derivative of ϕ is

$$
\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{1}{N\sinh t}\sum_q\Big((b_q\dot{P}_q + b_q^\dagger\dot{P}_q^*) - \coth t(b_qP_q + b_q^\dagger P_q^*)\Big) \tag{5.9}
$$

The derivatives $\dot{P}_q, \dot{P}_q^*$ are not easily expressed in terms of P_q, P_q^*, so already this is an indication that any Hamiltonian that generates the time evolution must be quite complicated (or at least much more complicated than our proposed $\mathcal{H}$). Nonetheless, we might hope that perhaps the situation simplifies at late times, i.e. $t\to\infty$, because in this limit dS is effectively static as shown in Section 2.1. The Legendre functions become approximately

$$
P^{\pm ip}_{\nu-1/2}(\cosh t) \approx \frac{2^{\mp ip}i^{-(\nu-1/2)}\Gamma(\pm 2ip+1)}{\Gamma(\pm ip - \nu + 3/2)\Gamma(\pm ip+1)}e^{\pm ipt} = A_p^{\pm}e^{\pm ipt} \tag{5.10}
$$

The $1/\sinh t$ factor simply becomes e^{-t}, so the time derivative is

$$
\frac{\mathrm{d}\phi}{\mathrm{d}t} \approx \frac{1}{Ne^t}\sum_q\big((ip-1)b_qP_q - (ip+1)b_q^\dagger P_q^*\big) \tag{5.11}
$$

We may observe that still $\dot{\phi}\neq i[\mathcal{H},\phi]$, even in the $t\to\infty$ limit.

One might think that we could solve our problems by instead taking $\mathcal{H}\propto\sum_q b_q^\dagger b_q$. Then at least in the late time limit we would have the desired time evolution. However, something else goes wrong: the density matrix would no longer be diagonal in the n index, so we would not have thermal behavior in each mode, much less a uniform temperature across all modes.

For general t, we expected to find $\dot{\phi}\neq i[\mathcal{H},\phi]$ because we did not have a timelike Killing vector. On the other hand, we saw earlier that for $t\to\pm\infty$ the metric (2.12) is static, in which case ∂_t becomes a Killing vector. Why, then, are we not able to match a dynamical Hamiltonian with the temperature in this limit? One explanation is that the dS temperature, and hence the observed particle creation, arises in part from time evolution itself, not just from lack of information beyond the horizon. Even though the spacetime is static at $t\to\pm\infty$, particles have been created in the intermediate period, which leads to the discrepancy seen above.

6 Conclusion

We have demonstrated that, while we are able to recover the de Sitter temperature by constructing a Hamiltonian mode-by-mode, the resulting entanglement Hamiltonian does not correspond to the time coordinate of the hyperbolic slicing. Though one might have anticipated this result, it presents a puzzling question for our physical intuition, namely, how do we interpret the dS temperature? To be more concrete: an observer sitting in dS sees thermal radiation at a given time slice, but what would he or she see as time progresses?

The analysis in Sections 4 and 5 uses the hyperbolic slicing of dS, which, as emphasized many times, does not possess a timelike Killing vector. It would be interesting to repeat the analysis in the static coordinates, since in that case we would have a timelike Killing vector in each causal patch, and see if the resulting entanglement Hamiltonian can be shown to be consistent with time evolution in the static patch.

There is also the issue of what exactly the entanglement entropy of dS corresponds to. Based on our results, we would not be able to conclude that the entanglement entropy accounts entirely for the gravitational entropy of the horizon, seeing as our computed entropy does not seem to scale with the area of the dS horizon. There must be some aspect of gravitational entropy that cannot be described by our quantum-field-theoretic model. This leaves us with the question: what, then, are the states enumerated by gravitational entropy?

de Sitter space remains as perplexing and fascinating as ever, and the aforementioned questions may lead to interesting directions for future research. Despite not having all the answers, we have achieved a deeper understanding of the subtleties of quantum field theory in dS, especially those arising from its nonstatic nature. It is small steps like these that further our understanding of complicated cosmological spacetimes and, by extension, our own universe.

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