MAT 92 final paper:
The intersection form on 4-manifolds

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In this paper we prove some properties of the intersection form. We also state a few important theorems and examples.

1 Definitions

Let M be a compact, orientable, path-connected 4-manifold. Choose a generator $[M]$ of $H_4(M;\mathbb{Z})$. This is equivalent to choosing an orientation of M. Let $\smile$ denote the cup product on $H^*(M;\mathbb{Z})$ and $\frown$ denote the cap product on $H^k(M;\mathbb{Z}) \times H_\ell(M;\mathbb{Z}) \to H_{k-\ell}(M;\mathbb{Z})$. Let $\langle \cdot, \cdot \rangle$ denote the Kronecker pairing on $H^k(M;\mathbb{Z}) \times H_k(M;\mathbb{Z}) \to \mathbb{Z}$.

We define the intersection form $Q : H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) \to \mathbb{Z}$ by

$$Q(\alpha, \beta) = \langle \alpha \smile \beta, [M] \rangle \tag{1.1}$$

$\forall\ \alpha, \beta \in H^2(M;\mathbb{Z})$.

2 Properties

There are some properties that are easy to see. First, Q is bilinear since $\smile$ and $\langle \cdot, \cdot \rangle$ are bilinear. Moreover, since $\alpha \smile \beta = (-1)^{|\alpha||\beta|}(\beta \smile \alpha)$, $\smile$ is commutative for elements of $H^2(M;\mathbb{Z})$. Thus Q is symmetric. Q also vanishes on the torsion part of $H^2(M;\mathbb{Z})$ since if $n\alpha = 0$ then

$$0 = Q(n\alpha, \beta) = nQ(\alpha, \beta) \implies Q(\alpha, \beta) = 0 \tag{2.1}$$

by linearity, and same for if $n\beta = 0$. For the rest of this paper we will just assume $H^2(M;\mathbb{Z})$ has no torsion.

An important property is that Q is unimodular, i.e. its matrix with respect to a basis of $H^2(M;\mathbb{Z})$ has determinant ± 1. We can see this via the isomorphisms

$$H^2(M;\mathbb{Z}) \cong \operatorname{Hom}\big(H_2(M;\mathbb{Z}),\mathbb{Z}\big) \cong \operatorname{Hom}\big(H^2(M;\mathbb{Z}),\mathbb{Z}\big) \tag{2.2}$$

due to the universal coefficient theorem and Poincare duality. Namely, the first isomorphism is given by the map $h : H^2(M;\mathbb{Z}) \to \operatorname{Hom}\big(H_2(M;\mathbb{Z}),\mathbb{Z}\big)$ defined by

$$h(\alpha)(\gamma) = \langle \alpha, \gamma \rangle \tag{2.3}$$

$\forall\ \alpha \in H^2(M;\mathbb{Z}),\ \gamma \in H_2(M;\mathbb{Z})$, or in other words

$$h : \alpha \mapsto \langle \alpha, \cdot \rangle \tag{2.4}$$

$\forall\ \alpha \in H^2(M;\mathbb{Z})$. The second isomorphism is given by the map $D : H^2(M;\mathbb{Z}) \to H_2(M;\mathbb{Z})$ defined by

$$D(\alpha) = \alpha \frown [M] \tag{2.5}$$

$\forall\ \alpha \in H^2(M;\mathbb{Z})$. Now Q can be viewed as the composition of these maps in the following way. Q defines a map $H^2(M;\mathbb{Z}) \to \operatorname{Hom}\big(H^2(M;\mathbb{Z}),\mathbb{Z}\big)$ given by

$$\tilde{Q} : \alpha \mapsto \langle \cdot\,, D(\alpha) \rangle \tag{2.6}$$

since

$$\tilde{Q}(\alpha)(\beta) = \langle \beta, D(\alpha) \rangle = \langle \beta, \alpha \frown [M] \rangle = \langle \beta \smile \alpha, [M] \rangle = Q(\beta,\alpha) \tag{2.7}$$

Thus $\tilde{Q} : H^2(M;\mathbb{Z}) \to \operatorname{Hom}\big(H^2(M;\mathbb{Z}),\mathbb{Z}\big)$ is an isomorphism, so Q is nondegenerate. Moreover, since ± 1 are the only units in $\mathbb{Z}$, Q is unimodular.

Q also has the property that the intersection form of a connected sum $M\#N$ of 4-manifolds M, N is the direct sum of the intersection forms of each manifold. We can see this by observing that deleting a 4-ball from each manifold does not change its 2-dimensional homology, so the connected sum operation just gives us the direct sum of the 2-dimensional homology groups.

3 Simple examples

The simplest example is S^4, which has trivial intersection form since its 2-dimensional cohomology is trivial.

A more interesting example is $\mathbb{CP}^2$, which has $H^2(\mathbb{CP}^2;\mathbb{Z}) = \mathbb{Z}$. Thus $Q_{\mathbb{CP}^2} = \pm 1$ are the only possibilities, and we see that both are realized by choosing different orientations for $\mathbb{CP}^2$. In general for any M, choosing the opposite orientation class just flips the sign of Q.

We can also consider $S^2 \times S^2$, which has $H^2(S^2 \times S^2;\mathbb{Z}) = \mathbb{Z}^2$. Let α, β be the generators. We see that $\alpha \smile \beta$ generates the 4-dimensional cohomology, and also $\alpha \smile \alpha = \beta \smile \beta = 0$, so if we choose the standard orientation then

$$Q_{S^2 \times S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.1}$$

We can get many more examples via the connected sum operation.

4 Theorems

First let us define the following notion: we say that Q is even if $Q(\alpha, \alpha) \in 2\mathbb{Z}\ \forall$ basis elements α. Otherwise we say Q is odd.

A nice lemma states that if the 2nd Stiefel-Whitney class of M vanishes, i.e. $w_2(M) = 0$, then Q is even. In particular this applies to the case where M is spin.

The intersection form is very useful for classifying 4-manifolds. Freedman's theorem states that if Q is an even unimodular symmetric bilinear form over $\mathbb{Z}$ then there is a unique simply-connected topological 4-manifold M which has Q as its intersection form; if Q is odd, then there are two such manifolds, with at least one having no smooth structure.

Furthermore, Donaldson's theorem states that if M is a closed, orientable, smooth 4-manifold with positive (resp. negative) definite intersection form Q, then Q can be diagonalized to the identity (resp. negative identity) matrix over $\mathbb{Z}$.

5 The E_8 manifold

In light of the previous two theorems, we can demonstrate the existence of a closed, orientable, non-smoothable topological 4-manifold. Take the following 8×8 matrix:

$$E_8 = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix} \tag{5.1}$$

One can check by direct calculation that this matrix is unimodular, negative definite, and even. In particular since E_8 is even, it is not diagonalizable to the negative identity matrix. Thus from Freedman's theorem there exists a closed, oriented topological 4-manifold M with this intersection form, and from Donaldson's theorem M does not admit a smooth structure.