Incompleteness theorems in cosmology

Loki Lin

April 2024

1 Introduction to cosmology

In class we wrote down the Einstein equations as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0\,. \tag{1.1}$$

There is a generalization of this which describes cosmology:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0\,. \tag{1.2}$$

Here, $\Lambda \in \mathbb{R}$ is a free parameter known as the cosmological constant. When $\Lambda \neq 0$, solutions are non-static. Let us write down a non-static ansatz that is homogeneous and isotropic in space:

$$ds^2 = -dt^2 + a^2(t) d\vec{x}^2\,. \tag{1.3}$$

Note that this ansatz is invariant under space translations, i.e. $\vec{x} \mapsto \vec{x} + \vec{c}$, as well as rotations and reflections, i.e. $\vec{x} \mapsto R\vec{x}$ for any 3-by-3 matrix R satisfying $R^\top R = 1$. Such a spacetime is called homogeneous (translation-invariant) and isotropic (rotation-invariant) in *space* (not spacetime). This ansatz is not the most general possible – there are solutions which do not satisfy homogeneity and/or isotropy. We will discuss those later.

For now, we want to find the allowed behavior of $a(t)$ according to (1.2). Spherical symmetry dictates that there are three families of spatial metrics:

$$d\vec{x}^2 = \frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\,, \tag{1.4}$$

where $d\Omega^2$ is the usual metric on S^2 and $\kappa \in \{0, +1, -1\}$ since if $\kappa \neq 0$ then we can always rescale r to make $|\kappa| = 1$. The three choices of κ correspond to flat, closed (positively curved), and open (negatively curved) respectively. Now the nonvanishing components of the Ricci tensor are

$$R_{\mu\nu} = \begin{pmatrix} -3\frac{\ddot{a}}{a} & 0 & 0 & 0 \\ 0 & \frac{\ddot{a}a + 2\dot{a}^2 + 2\kappa}{1-\kappa r^2} & 0 & 0 \\ 0 & 0 & r^2(\ddot{a}a + 2\dot{a}^2 + 2\kappa) & 0 \\ 0 & 0 & 0 & r^2(\ddot{a}a + 2\dot{a}^2 + 2\kappa)\sin^2\theta \end{pmatrix}. \tag{1.5}$$

Taking the trace, we find that the Ricci scalar is

$$R = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2}\right). \tag{1.6}$$

The tt-component of the field equations gives

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = 0\,, \tag{1.7}$$

while the rr-component gives

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = 0\,. \tag{1.8}$$

Together we may write these as

$$\begin{cases} \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = 0\,, \\ \frac{\ddot{a}}{a} = 0\,. \end{cases} \tag{1.9}$$

These are the Friedmann equations in vacuum. Often we work with the Hubble parameter (sometimes called the Hubble constant in physics literature, even though it has time dependence)

$$H = \frac{\dot{a}}{a}\,. \tag{1.10}$$

Let us pause for a moment to explain what $a(t)$ and $H(t)$ mean physically. If we pick an object that is some distance away from us at $t = 0$, $a(t)$ describes how it moves towards or away from us. On the other hand, if we pick two objects that start out with some separation at $t = 0$, $H(t)$ describes how much farther apart or closer together they are at time t. In particular,

$$\dot{H}(t) = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2, \tag{1.11}$$

so it is possible for $\dot{H}$ and $\ddot{a}$ to have different signs. Usually we say that the sign of $\ddot{a}$ corresponds to whether the expansion of the universe is accelerating or de-accelerating. The physical meaning of H will become important later on, when we generalize its definition to a non-homogeneous, non-isotropic universe.

2 Proof of incompleteness theorem

In this section we follow the proof of Guth et al. (2003) [2].

2.1 Homogeneous, isotropic universe

Let us consider a timelike observer $\mathcal{O}$ in the spacetime (1.3). If we write

$$P^\mu = \begin{pmatrix} E \\ \vec{p} \end{pmatrix} \tag{2.1}$$

for $\mathcal{O}$'s four-momentum, the energy-momentum relation gives

$$-E^2 + a^2(t)\, p^2 = -m^2 \,, \tag{2.2}$$

or

$$E(t) = \sqrt{p^2(t) + m^2} \,. \tag{2.3}$$

In particular, if we choose some reference time t_f and decompose the spatial momentum as

$$\vec{p}(t) = \frac{1}{a(t)}\, a(t_f)\, \vec{p}(t_f) \,, \tag{2.4}$$

then we have

$$E(t) = \sqrt{\frac{1}{a^2(t)}\, a^2(t_f)\, p^2(t_f) + m^2} \,. \tag{2.5}$$

The proper time along $\mathcal{O}$'s path is

$$d\tau^2 = dt^2 - a^2(t)\, d\vec{x}^2 \,. \tag{2.6}$$

Since

$$\frac{dt}{d\tau} = \frac{E}{m} \,, \tag{2.7}$$

we have that

$$d\tau = \frac{m}{E} dt \,. \tag{2.8}$$

Now consider the integral

$$H_{\rm av} \Delta\tau = \int_{\tau_i}^{\tau_f} H(\tau)\, d\tau \,. \tag{2.9}$$

Changing variables:

$$H_{\rm av} \Delta\tau = \int_{t_i}^{t_f} H(t)\, \frac{m}{E}\, dt = \int_{a_i}^{a_f} \frac{m\, da}{aE} \,, \tag{2.10}$$

where we are using the shorthand $a(t_f) = a_f$ and $a(t_i) = a_i$. Plugging in (2.5) gives

$$H_{\rm av} \Delta\tau = \int_{a_i}^{a_f} \frac{m\, da}{\sqrt{a_f^2 p_f^2 + m^2 a^2}} \,, \tag{2.11}$$

where we also abbreviate $\vec{p}(t_f) = \vec{p}_f$. Now if $\dot{a}, a > 0$, we can upper-bound this integral by changing the lower limit to $a = 0$. Thus we have

$$H_{\rm av} \Delta\tau \leq \int_0^{a_f} \frac{m\, da}{\sqrt{a_f^2 p_f^2 + m^2 a^2}} \tag{2.12}$$

$$= \frac{1}{2} \ln \left(\frac{E_f + m}{E_f - m} \right) = U \,, \tag{2.13}$$

where the final energy is $E_f = \sqrt{p_f^2 + m^2}$. Now the "average" Hubble parameter is upper-bounded by

$$H_{\rm av} \le \frac{U}{\Delta\tau} \,, \tag{2.14}$$

and in particular if $H_{\rm av} > 0$ along the geodesic, then the proper time along the geodesic is also upper-bounded:

$$\Delta\tau \le \frac{U}{H_{\rm av}} \,. \tag{2.15}$$

Thus, any timelike geodesic necessarily only goes on for a finite amount of proper time.

We can prove a similar result for null geodesics. If $\mathcal{O}$ is travelling along a null geodesic, then we have

$$\frac{d^2 t}{d\lambda^2} + \frac{\dot a}{a} \left(\frac{dt}{d\lambda} \right)^2 = 0 \,, \tag{2.16}$$

Solving this yields

$$d\lambda \propto a(t)\, dt \,. \tag{2.17}$$

To keep λ dimensionless, let us choose

$$d\lambda = \frac{a(t)}{a(t_f)}\, dt \,. \tag{2.18}$$

Now consider the integral

$$H_{\rm av} \Delta\lambda = \int_{\lambda_i}^{\lambda_f} H(\lambda)\, d\lambda \,. \tag{2.19}$$

Changing variables:

$$H_{\rm av} \Delta\lambda = \int_{t_i}^{t_f} H(t)\, \frac{a(t)}{a(t_f)}\, dt = \int_{a_i}^{a_f} \frac{da}{a(t_f)} \,. \tag{2.20}$$

Again we can upper-bound this by changing the lower limit to $a = 0$. Thus we have

$$H_{\rm av} \Delta\lambda \le 1 \,, \tag{2.21}$$

which implies

$$\Delta\lambda \le \frac{1}{H_{\rm av}} \,, \tag{2.22}$$

as long as $H_{\rm av} > 0$ along the path. Hence, we see that null geodesics are also forced to have a finite affine length.

To summarize, we have shown that if we pick any reference time t_f and consider a *causal* geodesic that goes backwards in time, the geodesic necessarily ends in a finite amount of proper time or affine length. We say that causal geodesics are past-incomplete, or the universe is past-incomplete. Our proof relies on the fact that the averaged Hubble parameter satisfies $H_{\rm av} > 0$.

2.2 General case

We now move to the case where the universe is not necessarily homogeneous or isotropic. The key to the proof is to define a generalized version of the Hubble parameter. We will see that, for a suitable definition of $H(t)$, all we will assume is the same condition as before, $H_{\rm av} > 0$ along a causal geodesic. We have no need to write down an ansatz for the metric at all.

To get a sense of how to define $H(t)$ in curved space, let us first review the meaning of $H(t)$ in flat space. Suppose we have an observer $\mathcal{O}$ moving along a timelike geodesic, and let us supply $\mathcal{O}$ with a continuous cascade of comoving test particles parameterized by the four-velocity $u^\mu(\tau)$. Fix two points in the path, τ_1 and τ_2. The separation between the test particles at those points is simply

$$\Delta\vec{r} = \vec{r}_1 - \vec{r}_2 = \vec{u}\Delta\tau\,. \tag{2.23}$$

In $\mathcal{O}$'s frame, the relative velocity between these particles is

$$\Delta\vec{u} = \vec{u}(\tau_1) - \vec{u}(\tau_2) = -\frac{d\vec{u}}{d\tau}\Delta\tau\,. \tag{2.24}$$

Recall that $H(t)$ describes how much farther apart two particles have moved after time t. We can define

$$H = \frac{(\Delta\vec{u})_r}{|\Delta\vec{r}|}\,. \tag{2.25}$$

In terms of the space-separation between the particles, we have

$$H = \frac{\Delta\vec{u}\cdot\Delta\vec{r}}{|\Delta\vec{r}|^2}\,, \tag{2.26}$$

Plugging in (2.23) and (2.24) gives

$$H(\tau) = \frac{d\vec{u}}{d\tau}\cdot\frac{\Delta\vec{r}}{|\Delta\vec{r}|^2}\Delta\tau = \frac{d|\vec{u}|}{d\tau}\,. \tag{2.27}$$

In the case of curved space, we want to use the same definition (2.25), so we simply need to find $\Delta\vec{u}$ and $\Delta\vec{r}$. Let v^μ be the four-velocity of $\mathcal{O}$, so that

$$\gamma(\tau) = u_\mu(\tau)\,v^\mu = \frac{1}{\sqrt{1-(\Delta v)^2}} \tag{2.28}$$

is the relative Lorentz factor between $\mathcal{O}$ and each test particle.

We must be careful with Δr^μ: it is the component of $u^\mu\Delta\tau$ that is *orthogonal* to u^μ, because it should be measured at equal times in each particle's rest frame. Thus we have

$$\Delta r^\mu = v^\mu\Delta\tau - \gamma u^\mu\Delta\tau\,, \tag{2.29}$$

and

$$|\Delta r| = \sqrt{1-\gamma^2}\Delta\tau\,. \tag{2.30}$$

where we have used $u^2 = -1$. At this stage, we can check that in the nonrelativistic limit, $\gamma \to 1$ so

$$\Delta r^\mu = \begin{pmatrix} 0 \\ \vec{u}\Delta\tau \end{pmatrix}, \tag{2.31}$$

in agreement with our analysis for flat space. The generalization for the relative velocity is simple: we replace $\frac{d}{d\tau}$ with the covariant derivative along $\mathcal{O}$, i.e.

$$\Delta u^\mu = \frac{Du^\mu}{D\tau}\Delta\tau\,. \tag{2.32}$$

Now we are ready to proceed with the generalization of the Hubble parameter. Following (2.25), we have

$$H = \frac{\Delta u^\mu \Delta r_\mu}{|\Delta r|^2} = \frac{Du^\mu}{D\tau}\frac{\Delta r_\mu}{|\Delta\vec{r}|^2}\Delta\tau \tag{2.33}$$

Plugging in (2.30) gives

$$H = \frac{Du^\mu}{D\tau}\frac{1}{1-\gamma^2}v_\mu\,. \tag{2.34}$$

Now, note that

$$\frac{d\gamma}{d\tau} = \frac{D\gamma}{D\tau} = \frac{Du^\mu}{D\tau}v^\mu + u^\mu\frac{Dv_\mu}{D\tau}\,, \tag{2.35}$$

and since v^μ is geodesic, we have

$$\frac{Du^\mu}{D\tau}v_\mu = \frac{d\gamma}{d\tau}\,. \tag{2.36}$$

Thus, (2.34) simplifies to

$$H = \frac{d\gamma}{d\tau}\frac{1}{1-\gamma^2} = \frac{d\gamma}{d\tau}F'(\gamma)\,, \quad F(\gamma) = \frac{1}{2}\ln\frac{\gamma+1}{\gamma-1}\,. \tag{2.37}$$

Finally, we define the averaged Hubble parameter as before:

$$H_{\text{av}} = \frac{1}{\Delta\tau}\int_{\tau_i}^{\tau_f} = \frac{F(\gamma_f)-F(\gamma_i)}{\Delta\tau}\,, \tag{2.38}$$

where $\gamma_f = \gamma(\tau_f)$ and $\gamma_i = \gamma(\tau_i)$. If $F(\gamma_i) \geq 0$, then H_{av} is upper bounded by

$$H_{\text{av}} \leq \frac{F(\gamma_f)}{\Delta\tau}\,, \tag{2.39}$$

and if we assume that $H_{\text{av}} > 0$ along the geodesic, then the proper time is upper-bounded by

$$\Delta\tau \leq \frac{F(\gamma_f)}{H_{\text{av}}}\,, \tag{2.40}$$

which is finite and reduces to (2.15) if we use $t(\tau_f)$ as the reference time.

We will not repeat the proof for null geodesics, but it can be shown in the same manner that any past-directed null geodesic terminates in a finite amount of affine length. Together, this shows the geodesic incompleteness of inflationary models. The condition $H_{\text{av}} > 0$ is often called the "averaged expansion condition."

3 References

1. Ed Witten's lectures in PHY 539: *Quantum Mechanics and Gravity* at Princeton in Fall 2022.

2. A. Borde, A. H. Guth, and A. Vilenkin, Phys. Rev. Lett. **90** (2003).