# Topological Invariants in (2+1)-Dimensional Chern-Simons Theory


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Independent Work

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19 April 2022


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## Abstract


Motivated by the Chern-Simons/Wess-Zumino-Witten correspondence, this paper studies Chern-Simons theory on a 3-manifold using mathematical gauge theory and path integral quantization. We review the necessary math and physics concepts, keeping an undergraduate reader in mind. We show that the Gauss linking number of a collection of particle trajectories in spacetime can be computed via the Wilson loop of the theory.


*This paper represents my work in accordance with University regulations.*

/s/ Loki Lin

# Contents

# 1 Introduction

The holographic principle is one of the greatest prospects of modern high-energy theory. For example, the anti-de Sitter/conformal field theory (AdS/CFT) correspondence gave a link between string theory in a bulk AdS spacetime and a CFT living on the boundary. Correspondences like this are helpful because they enable us to bypass tricky CFT calculations by computing a dual quantity in the bulk theory instead.

Much like how AdS string theory is dual to a boundary CFT, 3d Chern-Simons (CS) theory is dual to a 2d Wess-Zumino-Witten (WZW) model. This was known even before the development of AdS/CFT. In fact, it was Ed Witten's 1989 paper [1] on CS theory and the Jones polynomial that kicked off work on this correspondence. Unlike AdS/CFT, which is just a conjecture for now, the CS/WZW correspondence is a well-established theorem, with the CS side being described by topology and the WZW side being described by algebraic geometry [2, 3]. Equally exciting was the fact that Witten was able to formulate knot invariants as path integrals over moduli spaces of principal bundles.

This paper provides a review Sections 1 and 2 of Witten's 1989 paper at a level (hopefully) accessible to undergraduate math and physics majors. In particular, we hope to bridge the gap between what mathematicians and physicists might be comfortable with, and use the language of both sides. To that end, we begin in Section 2 with the mathematical formalism necessary to understand gauge theory from a pure math point of view. Next, in Section 3 we introduce the action of 3d Chern-Simons theory and prove some important characteristics. In Section 4 we formulate the gauge-invariant observables of the theory, which are 1d objects called Wilson loops. In Section 5 we quantize the theory using the path integral approach to quantum field theory and discuss subtleties such as gauge-fixing. In Section 6 we use all these tools to derive a correspondence between the Wilson loop and the Gauss linking number from knot theory. Finally we end in Section 7 with a summary of next steps and applications.

## 2 Bundles and connections

In this section we give a short overview of mathematical gauge theory following [4, 5]. It is usually the case that, when one learns gauge theory as an undergraduate physics major, the geometric structure is completely ignored. While, functionally speaking, it is often not necessary to know the underlying constructions in order to do field-theoretic calculations, for our purposes it will be essential.

Let $G$ be a simple Lie group and $M$ a compact oriented smooth manifold. A smooth principal $G$-bundle is a locally trivial fiber bundle $P$ over $M$ with standard fiber $G$ such that the projection map $\pi : P \to M$ is smooth and $G$ acts freely on $P$. $G$ is called the structure group or the gauge group. Given any representation $\rho : G \to \mathrm{GL}(V)$ of $G$, we have a group action of $G$ on $P \times V$ given by

$$g(p, v) = \big(gp, \rho(g^{-1})v\big) \tag{2.1}$$

We define the associated vector bundle over $M$ as $E = (P \times V)/G$ (i.e. the quotient of $P \times V$ by the action of $G$) with the projection map $\overline{\pi} : \overline{(p, v)} \mapsto \pi(p)$. We denote by $\Gamma(E)$ the space of smooth sections of $E$ and $\mathrm{Vect}(M)$ the space of vector fields on $M$.

A connection on $E$ is a linear map $D : \mathrm{Vect}(M) \to \mathrm{End}\big(\Gamma(E)\big)$ taking $v \mapsto D_v$ such that $D_v$ is linear over $\Gamma(E)$ and satisfies the Leibniz rule $D_v(fs) = v(f)s + fD_v s$ for smooth functions $f \in C^\infty(M)$ and sections $s \in \Gamma(E)$. $D_v$ is called the covariant derivative in the direction of $v$. Given local coordinates $x^\mu$ on a neighborhood $U$ in $M$ and the associated basis of vector fields $\partial_\mu$, we write $D_\mu = D_{\partial_\mu}$. In this coordinate neighborhood we can also take a basis $e^i$ $(i = 1, \dots, \dim(V))$ of sections of $E$ since $E$ is locally trivial, and then we can write any section $s$ as $s = s^i e_i$. We define functions $A^i_{\mu j} : U \to \mathfrak{g}$ by $D_\mu e_j = A^i_{\mu j} e_i$. The covariant derivative is related to the partial derivative by the following:

$$(D_\mu s)^i = \partial_\mu s^i + A^i_{\mu j}(s^j) = \partial_\mu s^i + A^a_\mu T^{ai}_j(s^j) \tag{2.2}$$

where $T^a$ $(a = 1, \dots, \dim(\mathfrak{g}))$ are the generators of $\mathfrak{g}$ in the representation $\rho$ of $G$. More simply, we can write $D_\mu = \partial_\mu + A_\mu$. In physics, $A_\mu$ is sometimes called the connection, or more commonly the gauge field.

Given some connection on $E$, we define the curvature $F : \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to$

$\mathrm{End}\big(\Gamma(E)\big)$ by

$$F(v,w) = [D_v, D_w] - D_{[v,w]} \tag{2.3}$$

where $[v,w]$ is the Lie derivative, $[v,w](p) = (\partial_v w - \partial_w v)(p)$.

Again working in local coordinates, we write $F_{\mu\nu} = F(\partial_\mu, \partial_\nu) = F^j_{\mu\nu i}\, e_j \otimes e^i$ where $e_j$ is the dual basis corresponding to the basis of sections $e^i$. Computing the components explicitly gives the formula

$$F^j_{\mu\nu i} = \partial_\mu A^j_{\nu i} - \partial_\nu A^j_{\mu i} + A^j_{\mu k} A^k_{\nu i} - A^i_{\nu k} A_{\mu i} \tag{2.4}$$

or more compactly

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{2.5}$$

This is the so-called field strength tensor in physics. It is also useful to define the curvature two-form $F$ as

$$F = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu = dA + A \wedge A \tag{2.6}$$

where $A = A_\mu\, dx^\mu$.

We now turn to the notion of a gauge transformation on $E$, which is a bundle automorphism $g : E \to E$ that commutes with the projection map. The set of all such transformations is denoted $\mathcal{G}$, and it forms a group under pointwise multiplication. $\mathcal{G}$ is also called the gauge group, and it does contain $G$, but it is much larger than $G$ because not all gauge transformations are connected to the identity. A gauge transformation can be viewed as a transformation of the connection, which we write as $A_\mu \mapsto gA_\mu g^{-1} + g\partial_\mu g^{-1}$ for some $g \in \mathcal{G}$. Physical theories must be invariant under this $\mathcal{G}$-action. Thus we should consider what sorts of gauge-invariant quantities there are. The curvature transforms as $F_{\mu\nu} \mapsto gF_{\mu\nu}g^{-1}$, so it is not invariant. However, due to the cyclic property of trace, the quantity $\mathrm{Tr}(F \wedge \star F)$ is invariant, and so are $\mathrm{Tr}(\bigwedge_{i=1}^n F)$ for any $n$.

To get scalars, we integrate these forms over $M$, so the dimension of $M$ further restricts the types of terms we can have. Notably, the Yang-Mills action is given by

$$S_{\mathrm{YM}} = -\frac{1}{2} \int_M \mathrm{Tr}(F \wedge \star F) \tag{2.7}$$

This action exists for a manifold of any dimension. It is also worth noting that, if $M$ is $2n$-dimensional, we have

$$\left(\frac{i}{2\pi}\right)^n \frac{1}{n!} \int_M \mathrm{Tr}(F^n) \in \mathbb{Z} \tag{2.8}$$

where $F^n = \bigwedge_{i=1}^n F$. The deRham cohomology class of $\left(\frac{i}{2\pi}\right)^k \frac{1}{k!} \mathrm{Tr}(F^k)$ is known as the $k$th Chern class of $E$ and is denoted $c_k(E)$.

## 3 The Chern-Simons action

This section draws from [1, 4, 6].

A less obvious choice of action is the Chern-Simons (CS) action

$$S_{\mathrm{CS}} = \frac{k}{4\pi} \int_M \mathrm{Tr}\big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\big) \tag{3.1}$$

where $\dim(M) = 3$. We can rewrite this in local coordinates as

$$S_{\mathrm{CS}} = \frac{k}{8\pi} \int_M \epsilon^{\mu\nu\rho}\, \mathrm{Tr}\big(A_\mu(\partial_\nu A_\rho - \partial_\rho A_\nu) + \frac{2}{3} A_\mu [A_\nu, A_\rho]\big) \tag{3.2}$$

Since $S_{\mathrm{CS}}$ is not dependent on a choice of metric on $M$, it is a topological invariant of the $G$-bundle over $M$. This sets it apart from the usual Yang-Mills action because the definition of the Hodge star operator involves a metric.

There are a number of things to check. First we want to show that $S_{\mathrm{CS}}$ is invariant under gauge transformations that are connected to the identity in $\mathcal{G}$. Let us compute the variation in the action under a variation $A \mapsto A + \delta A$.

$$\begin{aligned}
\delta S_{\mathrm{CS}} &= \frac{k}{4\pi} \int_M \mathrm{Tr}\big(\delta A \wedge dA - A \wedge d(\delta A) + \frac{2}{3}(\delta A \wedge A \wedge A + A \wedge \delta A \wedge A + A \wedge A \wedge \delta A)\big) \\
&= \frac{k}{4\pi} \int_M \mathrm{Tr}\big(\delta A \wedge dA - dA \wedge \delta A + 2\, \delta A \wedge A \wedge A\big) \\
&= \frac{k}{2\pi} \int_M \mathrm{Tr}\big(\delta A \wedge F\big) = \frac{k}{2\pi} \int_M \epsilon^{\mu\nu\rho}\, \mathrm{Tr}\big(\delta A_\mu F_{\nu\rho}\big)
\end{aligned} \tag{3.3}$$

Now if $\delta A_\mu$ has the form of a derivative such as $D_\mu \varepsilon$, we can integrate by parts in the action, and then $\delta S_{\mathrm{CS}} = 0$ follows from the Bianchi identity $\epsilon^{\mu\nu\rho} D_\mu F_{\nu\rho} = 0$ in this case.

This calculation also shows that the classical equation of motion of the theory is

$F = 0$, so $A$ must be a flat connection. Notably, if $A$ is "pure gauge", i.e. $A = g^{-1}dg$ for some $g : M \to G$, then $F = 0$, and the converse is true locally.

We are more interested in how $S_{\text{CS}}$ behaves under gauge transformations that are not connected to the identity. Any gauge transformation can be written as $A \mapsto gAg^{-1} + g\,dg^{-1}$. Note that the action can be rewritten as

$$S_{\text{CS}} = \frac{k}{4\pi}\int_M \text{Tr}\big(A \wedge F - \frac{1}{3}A \wedge A \wedge A\big) \tag{3.4}$$

$F$ transforms as $F \mapsto gFg^{-1}$, so the variations of each term are

$$\delta\text{Tr}(A \wedge F) = \text{Tr}(dg^{-1} \wedge g\,dA + dg^{-1} \wedge gA \wedge A) \tag{3.5}$$

$$\delta\text{Tr}(A\wedge A\wedge A) = \text{Tr}(3dg^{-1}\wedge gA\wedge A + 3dg^{-1}\wedge g\,dg^{-1}\wedge gA + g\,dg^{-1}\wedge g\,dg^{-1}\wedge g\,dg^{-1}) \tag{3.6}$$

Finally, the variation in the action is

$$\delta S_{\text{CS}} = -\frac{k}{4\pi}\int_M \text{Tr}\big(d(dg^{-1} \wedge gA) + \frac{1}{3}g\,dg^{-1} \wedge g\,dg^{-1} \wedge g\,dg^{-1}\big) \tag{3.7}$$

The total derivative term can be ignored as usual, and the last term is related to an integer, called the "winding number" of the transformation, which depends only on the homotopy class of $g$.

$$w(g) = \frac{1}{24\pi^2}\int_M \text{Tr}\big(g\,dg^{-1} \wedge g\,dg^{-1} \wedge g\,dg^{-1}\big) \in \mathbb{Z} \tag{3.8}$$

In order to check that this is homotopy invariant, we can compute its variation with respect to $g \mapsto \delta g$. First let us rewrite it as

$$w(g) = -\frac{1}{24\pi^2}\int_M \text{Tr}\big(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\big) \tag{3.9}$$

and now the variation is

$$\delta w = -\frac{1}{8\pi^2}\int_M \text{Tr}\big(\delta(g^{-1}dg) \wedge g^{-1}dg \wedge g^{-1}dg\big)$$

$$= -\frac{1}{8\pi^2}\int_M \text{Tr}\Big(\big(-g^{-1}\delta g\,g^{-1}dg + g^{-1}d(\delta g)\big) \wedge g^{-1}dg \wedge g^{-1}dg\Big)$$

$$= \frac{1}{8\pi^2} \int_M d\,\mathrm{Tr}\big(g^{-1}\delta g\, d(g^{-1}dg)\big) \tag{3.10}$$

which vanishes by Stokes' theorem.

The fact that $w(g) \in \mathbb{Z}$ is more subtle. Let us consider the case $G = \mathrm{SU}(2)$ with the generators given by the Pauli matrices $\sigma_i$. Since SU(2) is homeomorphic to $S^3$, let us take the usual $\theta_i$ ($i = 1, 2, 3$) parametrization of $S^3$ where $\theta_1 \in [0, 2\pi]$, $\theta_2 \in [0, \pi]$, $\theta_3 \in [0, 4\pi]$, and identify an element of SU(2) as $f(\vec{\theta}) = e^{i\vec{\sigma}\cdot\vec{\theta}}$. Then we have

$$\int_{\mathrm{SU}(2)} \mathrm{Tr}\big(f^{-1}df \wedge f^{-1}df \wedge f^{-1}df\big) = -i \int_{S^3} \mathrm{Tr}\big((\vec{\sigma}\cdot d\vec{\theta}) \wedge (\vec{\sigma}\cdot d\vec{\theta}) \wedge (\vec{\sigma}\cdot d\vec{\theta})\big)$$

$$= 6 \int_{S^3} \mathrm{Tr}\mathbb{1}\, d\theta_1 \wedge d\theta_2 \wedge d\theta_3 = 24\pi^2 \tag{3.11}$$

Now let $g : M \to \mathrm{SU}(2)$ be a continuous map. $g$ induces a homomorphism on the homotopy groups, $g_* : \pi_3(M) \to \pi_3(\mathrm{SU}(2))$. Since $M$ is oriented, $\pi_3(M) \cong \pi_3(\mathrm{SU}(2)) \cong \mathbb{Z}$, so $g_*$ must be multiplication by an integer, $\deg(g)$. Finally, we can use the property that the induced maps on chains and forms are dual to each other with respect to the integration pairing, i.e. $\big\langle g_*[c], [\omega]\big\rangle = \big\langle [c], g^*[\omega]\big\rangle$ where $\big\langle [c], [\omega]\big\rangle = \int_c \omega$ for any singular chain $c$ and differential form $\omega$. In particular, letting $[M], [S^3]$ be generators of $\pi^3(M), \pi^3(S^3)$ respectively, we have

$$\int_M g^*\omega = \big\langle [M], g^*[\omega]\big\rangle = \big\langle g_*[M], [\omega]\big\rangle = \deg(g)\big\langle [S^3], [\omega]\big\rangle = \deg(g) \int_{S^3} \omega \tag{3.12}$$

We also have

$$g^*(f^{-1}df \wedge f^{-1}df \wedge f^{-1}df) = (g^*f^{-1})\big(d(g^*f)\big) \wedge (g^*f^{-1})\big(d(g^*f)\big) \wedge (g^*f^{-1})\big(d(g^*f)\big)$$

$$= g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg \tag{3.13}$$

Putting the pieces together, we conclude that

$$w(g) = -\frac{1}{24\pi^2} \int_M \mathrm{Tr}\big(f^{-1}df \wedge f^{-1}df \wedge f^{-1}df\big) = -\deg(g) \in \mathbb{Z} \tag{3.14}$$

To generalize to any simple Lie group $G$, we use a theorem from Lie theory which states that any map $f : S^3 \to G$ is homotopic to a map $\tilde{f} : S^3 \to \mathrm{SU}(2) \subset G$. This implies $w(g) \in \mathbb{Z}$ for $M = S^3$. For any compact oriented $M$, we again have that maps from $M$ to $S^3$ are classified by $\pi_3(M) \cong \mathbb{Z}$, so we conclude that the result holds for any compact oriented 3-manifold and simple Lie group.

We note that when $A$ is pure gauge, $S_{\mathrm{CS}}$ has a similar form to the Wess-Zumino-Witten (WZW) action. Namely, the $A \wedge A \wedge A$ term is the WZW term. The full WZW action is

$$S_{\mathrm{WZW}} = \frac{k}{4\pi} \int_{\partial M} \mathrm{Tr}\big(\partial^\mu g^{-1} \partial_\mu g\big) d^2x - \frac{k}{24\pi^2} \int_M \epsilon^{\mu\nu\rho} \, \mathrm{Tr}\big(g^{-1}\partial_\mu g \, g^{-1}\partial_\nu g \, g^{-1}\partial_\rho g\big) d^3x \quad (3.15)$$

for $M$ a manifold with boundary. The $S_{\mathrm{WZW}}$ action describes a two-dimensional theory in the sense that the classical equations of motion only depend on $\partial M$. This is because the variation of the WZW term can be written as a total derivative, as shown previously.

In quantum field theory, the action appears in the path integral as $e^{iS}$, which is invariant under $S \mapsto S + 2\pi n$ for $n \in \mathbb{Z}$. Thus the statement that physical theories should be invariant under all gauge transformations requires $k \in \mathbb{Z}$.

# 4 Wilson operators

This section makes use of [4].

In order to discuss Wilson lines and Wilson loops, we first need to define the notion of path ordering. Suppose we have a smooth path $\gamma : [0, 1] \to M$. Given a collection of points $t_1, \ldots, t_n \in [0, 1]$ and a connection form $A$, we define the path-ordered product as

$$\mathcal{P} A\big(\gamma'(t_1)\big) \times \ldots \times A\big(\gamma'(t_n)\big) = A\big(\gamma'(t_{\sigma(1)})\big) \times \ldots \times A\big(\gamma'(t_{\sigma(n)})\big) \quad (4.1)$$

where $\sigma$ is a permutation of $1, \ldots, n$ such that $t_{\sigma(j)} \geq t_{\sigma(j+1)}$. $\mathcal{P}$ is called the path-ordering operator. We can use this to define the path-ordered exponential

$$\mathcal{P} \exp\Big( - \int_0^t A\big(\gamma'(s)\big) ds \Big) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathcal{P} \Big( \int_0^t A\big(\gamma'(s)\big) \Big)^n \quad (4.2)$$

If $\gamma$ is a closed loop, we define the Wilson loop along $\gamma$ as

$$W(\gamma) = \mathrm{Tr}\Big[\mathcal{P}\exp\Big(-\int_0^1 A\big(\gamma'(s)\big)ds\Big)\Big] = \mathrm{Tr}\Big[\mathcal{P}\exp\Big(-\int_\gamma A_\mu\, dx^\mu\Big)\Big] \tag{4.3}$$

Wilson loops have a nice geometric interpretation which involves the notion of parallel transport. A section $s$ of $E$ along a path $\gamma : [0,1] \to M$ is said to be parallel to $\gamma$ with respect to a connection $D$ if $D_{\gamma'(t)}\big(s(t)\big) = 0$ for all $t \in [0,1]$, where $s(t) = s\big(\gamma(t)\big)$. Given an element $s_0$ of the fiber over $\gamma(0) \in M$, we define the parallel transport of $s_0$ along $\gamma$ as the solution to the ordinary differential equation $D_{\gamma'(t)}\big(s(t)\big) = 0$ with initial condition $s\big(\gamma(0)\big) = s_0$.

Now we can define an operator on the fiber $E_{\gamma(0)}$ called the holonomy $H(\gamma, D)$ : $E_{\gamma(0)} \to E_{\gamma(1)}$ by $H(\gamma, D)s\big(\gamma(0)\big) = s\big(\gamma(1)\big)$, where $s$ is the parallel transport of $s(p)$ along $\gamma$. When $\gamma(0) = \gamma(1) = p$, $H(\gamma, D) \in \mathrm{End}(E_p)$, so $H(\gamma, D)$ is an element of $G$ in the representation $\rho$. The Wilson loop along $\gamma$ is then simply the trace of $H(\gamma, D)$.

It is easier to understand the gauge invariance of $W$ using this parallel transport definition. Consider again the parallel transport equation $D_{\gamma'(t)}\big(s(t)\big) = 0$ and apply a gauge transformation to $s$ so that $\tilde{s}(t) = g(t)s(t)$ and $\tilde{A} = gAg^{-1} + g\, dg^{-1}$. Taking local coordinates on $M$, we compute

$$\begin{aligned}\tilde{D}_{\gamma'(t)}\tilde{s}(t) &= \tilde{s}'(t) + \gamma'^{\mu}(t)\tilde{A}_\mu\tilde{s}(t) = g'(t)s(t) + g(t)s'(t) + \gamma'^{\mu}(t)\tilde{A}_\mu\tilde{s}(t) \\ &= -\gamma'^{\mu}(t)\big(g\,\partial_\mu g^{-1} + gA_\mu g^{-1} - \tilde{A}_\mu\big)\tilde{s}(t) = 0\end{aligned} \tag{4.4}$$

Hence the holonomy transforms as

$$H(\gamma, D') = g(1)H(\gamma, D')g(0)^{-1} \tag{4.5}$$

and now if we set $\gamma(0) = \gamma(1)$ then trace cyclicity implies that $W(\gamma)$ is invariant under this transformation.

# 5 The path integral

The remainder of this paper follows [1] and occasionally makes use of [7] for background information.

To quantize Chern-Simons theory, we consider a functional integral over $\mathcal{A}$, the space of connections modulo gauge transformations. The Lebesgue integration measure on this space is denoted $\mathcal{D}A$. One can view the measure $\mathcal{D}A$ as a product $\prod_{x,\mu} dA_\mu(x)$ in some local coordinates; this can be shown to be gauge-invariant. We consider also a set of $r$ disjoint loops $C_i : [0,1] \to M$ together forming a link $L$ in $M$, with each loop corresponding to a representation $\rho_i$ of $G$. Then the partition function is

$$Z(M,L) = \int \mathcal{D}A \, \exp\big(iS_{\mathrm{CS}}[A]\big) \prod_{i=1}^{r} W(C_i,\rho_i)[A] \tag{5.1}$$

where $W(C_i,\rho_i)$ denotes the Wilson loop along $C_i$ with the trace taken in the representation $\rho_i$.

Let us first consider the path integral in the large $k$ limit, and without any Wilson loops. The main idea of path integrals, in general, is to weight each classical "trajectory", or in this case the equivalence classes of connections, with a complex phase proportional to the classical action of the theory. Then most of these phases will cancel out, except for contributions near the stationary points of the action. As noted previously, the stationary points of $S_{\mathrm{CS}}$ are given by the flat connections, i.e. those which are locally pure gauge. For now we can assume that there are a finite number of equivalence classes of flat connections. Let us take a representative $A^\alpha$ of each class. It is then useful to expand around these, writing more generally $A = A^{(\alpha)} + B$, so that the leading order terms in the action are

$$S_{\mathrm{CS}}[A] = S_{\mathrm{CS}}[A^{(\alpha)}] + \frac{k}{4\pi}\int_M \mathrm{Tr}\big(B \wedge D^{(\alpha)}B\big) \tag{5.2}$$

where $D^{(\alpha)}$ denotes the exterior covariant derivative with respect to $A^{(\alpha)}$. Often we will drop the superscript on $D^{(\alpha)}$ when there is no ambiguity.

In order to compute path integrals, we need to fix a gauge, or in other words pick a representative from each equivalence class in $\mathcal{A}$. Let us fix the Lorentz-invariant gauge condition $D^\mu B_\mu = 0$. To enforce this condition in the path integral, we can insert a delta function $\delta(D^\mu B_\mu)$. However, in order to retain gauge invariance, we must integrate over gauge transformations. We have

$$1 = \int \mathcal{D}g \, \delta\big(G(B^g)\big) \det \frac{\delta G(B^g)}{\delta g} \tag{5.3}$$

where $B^g$ is the gauge transformation of $B$ by $g$ and $G(B)$ is some constraint function. Thus we need an extra term, $\det\left[\delta(D^\mu B^g_\mu)/\delta g\right]$. This term is easier to treat when we put it in the form of a Gaussian integral. For any finite-dimensional operator $\mathcal{O}$ we have the identity

$$\det \mathcal{O} = \int \exp(\bar{c}_m \mathcal{O}_{mn} c_n) \prod_n d\bar{c}_n \, dc_n \tag{5.4}$$

where $m, n = 1, \ldots, \dim(\mathcal{O})$. The $c_n$ are anticommuting variables with values in the vector space that $\mathcal{O}$ acts on. This can be generalized to infinite-dimensional operators by taking $c, \bar{c}$ to be functions with values in an anticommuting algebra. Thus we arrive at the gauge-fixed action

$$S[A] = S_{\mathrm{CS}}[A] + \int_M \mathrm{Tr}\big(\phi\, D^\mu B_\mu + \bar{c}\, D_\mu D^\mu c\big) \tag{5.5}$$

where we have introduced a new bosonic field $\phi$ which comes from rewriting $\delta(D^\mu B_\mu)$ in the Fourier representation. We can view $\phi$ as a Lagrange multiplier that enforces the gauge condition.

We would like to write the kinetic terms in the form of a Gaussian integral with some self-adjoint operator. The kinetic term for the ghosts is already in this form with the Laplacian operator $\Delta = D_\mu D^\mu$. To treat the bosons, let us define $L_-$ to be the restriction of the operator $\star D + D\star$ to forms of odd degree. Then if we consider the inner product $(\mu, \omega) = \int_M \star\mu \wedge \omega$, we have

$$\begin{pmatrix} (\phi, L_-\phi) & (\phi, L_- B) \\ (B, L_-\phi) & (B, L_- B) \end{pmatrix} = \int_M \begin{pmatrix} 0 & \star\phi \wedge D \star B \\ \star B \wedge D \star \phi & \star B \wedge \star D B \end{pmatrix} \tag{5.6}$$

since only the forms of degree 3 contribute to the integral over $M$. Then, integrating by parts and rescaling $\phi$, we can rewrite the action as

$$S_{\mathrm{CS}}[A] = S_{\mathrm{CS}}[A^{(\alpha)}] + \frac{k}{4\pi}\mathrm{Tr}\Big((b, L_- b) + \int_M \bar{c}\Delta c\Big) \tag{5.7}$$

where we have combined the bosonic fields into the vector $b = (\phi\ B)$. In the partition function these terms are just Gaussian integrals with self-adjoint operators, so we can simplify the points of stationary phase to

$$\mu(A^{(\alpha)}) = \exp\left(iS_{\text{CS}}[A^{(\alpha)}]\right)\frac{\det\Delta}{\sqrt{\det L_-}} \tag{5.8}$$

Note that this expression with the determinants only makes sense if the deRham cohomology of $M$ with respect to the flat connection $A^{(\alpha)}$ is trivial.

At large $k$, then, the partition function is dominated by the stationary phase terms, so we write

$$Z = \sum_\alpha \mu(A^{(\alpha)}) \tag{5.9}$$

Since in the process of defining these operators we have chosen a metric on $M$, we must now consider whether the partition function is indeed a topological invariant. The absolute value of $\det\Delta/\sqrt{\det L_-}$ can be reformulated as the Ray-Singer torsion $\tau(M, A^{(\alpha)})$, which is a well-known topological invariant of $A^{(\alpha)}$.

Let us study the phase of $\det\Delta/\sqrt{\det L_-}$. $\Delta$ is just the Laplacian on functions, which we know is real and positive, so it is the phase of $\sqrt{\det L_-}$ that we must treat more carefully. Recalling from earlier that the term in the path integral with $L_-$ is just a Gaussian, we can rewrite it as

$$\prod_i \int e^{i\lambda_i x_i^2}\frac{dx_i}{\sqrt{\pi}} \tag{5.10}$$

where $x_i$ is an orthonormal basis of eigenfunctions of $L_-$ and $\lambda_i$ are the corresponding eigenvalues. Doing these integrals explicitly gives

$$\prod_i \frac{1}{\sqrt{|\lambda_i|}} e^{i\pi\,\text{sign}\lambda_i/4} \tag{5.11}$$

Thus we find the overall phase factor $\frac{\pi}{4}\sum_i \text{sign}\lambda_i$. Because the space of functions is infinite-dimensional, this sum is not necessarily finite, so we need to introduce a regularized version, the $\eta$-invariant

$$\eta(A^{(\alpha)}) = \frac{1}{2}\lim_{s\to 0}\sum_i |\lambda_i|^{-s}\,\text{sign}\lambda_i \tag{5.12}$$

so that the phase factor is $\frac{\pi}{2}\eta(A^{(\alpha)})$. This is almost a topological invariant, but not quite; it turns out that $\eta(A^{(\alpha)})$ depends on $\eta(0)$, the $\eta$-invariant of the trivial connection, by the formula

$$\eta(A^{(\alpha)}) = \eta(0) + \frac{c_2(G)}{\pi} I(A^{(\alpha)}) \tag{5.13}$$

where $c_2(G)$ is the eigenvalue of the quadratic Casimir operator in the adjoint representation of $G$ with the normalization convention $c_2\big(\mathrm{SU}(N)\big) = 2N$, and $I(A^{(\alpha)})$ is the Chern-Simons invariant

$$I(A^{(\alpha)}) = \frac{1}{4\pi}\int_M \mathrm{Tr}\big(A^{(\alpha)} \wedge dA^{(\alpha)} + \frac{2}{3}A^{(\alpha)} \wedge A^{(\alpha)} \wedge A^{(\alpha)}\big) \tag{5.14}$$

We would like to slightly redefine the path integral to account for the effect of $\eta(0)$. To do so requires some topological machinery – namely, the formalism of spin structures, so let us digress for a moment to discuss this.

An orientable Riemannian 3-manifold $M$ has a principal SO(3) bundle given by its orthonormal frame bundle $F_{\mathrm{SO}}(M)$. A spin structure on $M$ is a principal Spin(3) bundle $S(M)$ over $M$ such that the following are true:

1. There is a double-covering map $p_S : S \to F_{\mathrm{SO}}$ such that $p_S(sg) = p_S(s)p(g)$, where $p : \mathrm{Spin}(3) \to \mathrm{SO}(3)$ is the standard double-covering map;
2. $\pi \circ p_S = \pi_S$, where $\pi$ and $\pi_S$ are the projection maps corresponding to the bundles $F(M)$ and $S(M)$ respectively.

We now make use of a few topological results. The first fact is that all oriented 3-manifolds admit a spin structure, which is classified up to isomorphism by $H^1(M;\mathbb{Z}_2)$. Let us choose a spin bundle on $M$ with Levi-Civita connection $\omega$. Then we can define the "gravitational" Chern-Simons invariant

$$I(g) = \frac{1}{4\pi}\int_M \mathrm{Tr}\big(\omega \wedge d\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega\big) \tag{5.15}$$

The next fact is that the quantity

$$\Omega = \eta(0) + \frac{\dim(G)}{12\pi} I(g) \tag{5.16}$$

is a topological invariant, in that it is not dependent on the choice of metric. However, $\Omega$ is not invariant under the choice of framing of $M$, i.e. trivialization of the tangent bundle. Framings on $M$ are classfied by maps $M \to \mathrm{SO}(3)$, which are in turn classified by the degree of the map and the element of $H^1(M;\mathbb{Z}_2)$ associated with the spin structure on $M$. These two ambiguities combined give rise to a shift in $I(g)$ by $2\pi s$ with $s \in \mathbb{Z}$. Thus, we can simply replace $\eta(0) \mapsto 2\,\Omega$ in the partition function to get rid of the metric dependence. The partition function, finally, is

$$Z = \sum_{\alpha} \tau(M, A^{(\alpha)}) \exp\left(iS_{\mathrm{CS}}[A^{(\alpha)}] + \pi\,\Omega + c_2(G) I(A^{(\alpha)})\right) \tag{5.17}$$

Noting that $S_{\mathrm{CS}}[A^{(\alpha)}] = kI(A^{(\alpha)})$, we can write this more suggestively as

$$Z = \sum_{\alpha} \tau(M, A^{(\alpha)}) \exp\left[i\left(k + \frac{c_2(G)}{2}\right) I(A^{(\alpha)}) + \pi\,\Omega\right] \tag{5.18}$$

This form of the partition function makes it more clear that our regularization scheme for the operator determinants takes $k \mapsto k + c_2(G)/2$, and correcting for the choice of framing gives an extra $\pi\,\Omega$ phase.

# 6 Knots in the abelian case

In the following we will continue working in the large $k$ limit, and we will take $G = \mathrm{U}(1)$ and $M = S^3$. Since $\mathrm{U}(1)$ is abelian, the Chern-Simons action is simply

$$S_{\mathrm{CS}} = \frac{k}{4\pi} \int_M \mathrm{Tr}\big(A \wedge dA\big) \tag{6.1}$$

Now suppose we have some disjoint circles $C_\alpha$ embedded in $M$. We assign each an integer $n_\alpha$ corresponding to an irrep of $\mathrm{U}(1)$ in the usual way, recalling that irreps of $\mathrm{U}(1)$ are $\rho_n(e^{i\theta}) = e^{in\theta}$ for any $n \in \mathbb{Z}$.

Let us first address the Wilson loop, which appears in the path integral. Since $A$ is $\mathfrak{u}(1)$-valued, we need not worry about path ordering, so the Wilson loop is

$$W(C_\alpha, n_\alpha) = \prod_{\alpha} \exp\left(in_\alpha \oint_{C_\alpha} A\right) \tag{6.2}$$

We are interested in the expectation value of $W$. Expanding to leading order in $A$, we have

$$\big\langle W(C_\alpha,n_\alpha)\big\rangle = \Big\langle \prod_\alpha \exp\Big(in_\alpha \oint_{C_\alpha} A\Big)\Big\rangle = \prod_{\alpha,\beta}\exp\Big[-\frac{n_\alpha n_\beta}{2}\oint_{C_\alpha} dx^\mu \oint_{C_\beta} dy^\nu \big\langle A_\mu(x)A_\nu(y)\big\rangle\Big] \tag{6.3}$$

To evaluate the two-point function $\big\langle A_\mu(x)A_\nu(0)\big\rangle$, let us first rewrite the action in local coordinates as

$$S_{\mathrm{CS}} = \frac{k}{4\pi}\int_M \epsilon^{\mu\nu\rho}\,\mathrm{Tr}\big(A_\mu\partial_\nu A_\rho\big) \tag{6.4}$$

The two-point function is $4\pi/k$ times the Green's function $G_{\mu\nu}(x)$ for the differential operator $\epsilon^{\mu\lambda\nu}\partial_\lambda$. In other words, we must solve the differential equation

$$\epsilon^{\mu\lambda\nu}\partial_\lambda G_{\mu\nu}(x) = \delta^{(3)}(x) \tag{6.5}$$

It is easiest to do so in Fourier space, where the equation becomes

$$\epsilon^{\mu\lambda\nu}p_\lambda G_{\mu\nu}(p) = -i \tag{6.6}$$

We can now read off

$$G_{\mu\nu}(p) = i\epsilon_{\mu\lambda\nu}\frac{1}{p_\lambda} = i\epsilon_{\mu\lambda\nu}\frac{p^\lambda}{|p|^2} \tag{6.7}$$

Transforming back to position space, we have

$$\begin{aligned} G_{\mu\nu}(x) = i\epsilon_{\mu\lambda\nu}\int \frac{d^3p}{(2\pi)^3}\,e^{ip\cdot x}\,\frac{p^\lambda}{|p|^2} &= -\frac{i}{(2\pi)^2}\epsilon_{\mu\lambda\nu}\,\partial^\lambda \int_0^\infty d|p| \int_0^\pi d\theta\,\sin\theta\, e^{i|p||x|\cos\theta} \\ &= \frac{i}{2\pi^2}\epsilon_{\mu\lambda\nu}\,\partial^\lambda\frac{1}{|x|}\int_0^\infty du\,\frac{\sin u}{u} = \frac{i}{4\pi}\epsilon_{\mu\lambda\nu}\,\partial^\lambda\frac{1}{|x|} = -\frac{i}{4\pi}\epsilon_{\mu\nu\rho}\frac{x^\rho}{|x|^3} \end{aligned} \tag{6.8}$$

Thus the two-point function is

$$\big\langle A_\mu(x)A_\nu(0)\big\rangle = -\frac{i}{k}\epsilon_{\mu\nu\rho}\frac{x^\rho}{|x|^3} \tag{6.9}$$

Plugging this into the Wilson loop, we finally obtain

$$\big\langle W(C_\alpha,n_\alpha)\big\rangle = \exp\Big(\frac{i}{2k}\sum_{a,b} n_\alpha n_\beta \oint_{C_\alpha} dx^\mu \oint_{C_\beta} dy^\nu \epsilon_{\mu\nu\rho}\frac{(x-y)^\rho}{|x-y|^3}\Big)$$

$$= \exp\left(\frac{2\pi i}{k} \sum_{\alpha,\beta} n_\alpha n_\beta \, \Phi(C_\alpha, C_\beta; \alpha \neq \beta)\right) \tag{6.10}$$

where $\Phi$ is the Gauss linking number from knot theory

$$\Phi(C_\alpha, C_\beta; \alpha \neq \beta) = \frac{1}{4\pi} \oint_{C_\alpha} dx^\mu \oint_{C_\beta} dy^\nu \, \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3} \tag{6.11}$$

This is a well-known knot invariant that counts the signed number of times $C_\alpha$ crosses $C_\beta$, so that $\Phi(C_\alpha, C_\beta) = -\Phi(C_\beta, C_\alpha)$.

In the non-abelian case, the only difference is that the propagator has additional index structure corresponding to the generators of $\mathfrak{g}$; this just amounts to

$$\left\langle A_\mu^a(x) A_\nu^b(0) \right\rangle = -\frac{i}{k} \epsilon_{\mu\nu\rho} \frac{x^\rho}{|x|^3} \delta^{ab} \tag{6.12}$$

where $a, b$ label the generators of $\mathfrak{g}$.

Let us briefly digress to discuss the linking integral, since it is not obvious how the integral counts crossings. Parametrize two disjoint knots as curves $x^\mu$, $y^\nu : S^1 \to S^3$. We can then define the Gauss map $\Gamma : S^1 \times S^1 \to S^2 \subset S^3$ by

$$\Gamma^\mu(s,t) = \frac{x^\mu(s) - y^\mu(t)}{|x(s) - y(t)|} \tag{6.13}$$

We can then recognize the integrand of the linking integral as the Jacobian of this map, up to the factor of $4\pi$ which accounts for the surface area of $S^2$. If we choose a suitable plane to "slice" $S^3$, such that we get a link diagram for the knots, we see that the Gauss map gives a vector orthogonal to the plane at each crossing, and the orientation of the vector is determined by which knot is on top. When integrating over the Jacobian of the Gauss map, the contributions from all points will cancel except exactly where the knots cross; at these crossings, the sign is positive or negative depending on which knot is on top in the diagram. From the link diagram perspective it remains to be shown that the overall sign of the linking number is independent of the plane chosen to obtain the diagram, but from the integral formulation it is clear that this is true.

There is a complication that arises when $a = b$, i.e. the self-linking number of an individual knot. We need to define a covariant way of computing the self-linking number that gives a finite answer, since the above integrand is singular at $x = y$. One method is as follows. Given a knot $C$, we specify a framing $v$ of the knot, which is a vector field that is normal to and nowhere-vanishing on $C$. Such a vector field is of course not unique, but one can study homotopy classes of such fields, which correspond to different ways of "twisting" the framing. The framing gives us another knot $C_v$ which is simply $C$ displaced along $v$ everywhere by the same amount, and now we can define $\Phi(C; v) = \Phi(C, C_v)$.

By adding twists to the framing of any of the $C_\alpha$, we can shift $\Phi(C_\alpha)$ by an arbitrary integer $t$. The corresponding change in the Wilson loop is the addition of a phase $2\pi n_\alpha^2 t/k$.

# 7 Conclusion

Our story stops here, but there is a long road ahead. We have discussed a type of quantization called path integral quantization, but to proceed it will be necessary to understand how to quantize CS theory via canonical quantization. This is detailed in the remainder of Witten's paper and requires a little bit of complex geometry.

We have already been able to derive a topological invariant – the linking number – from observables in the physical theory. This can be done for other knot invariants too, most famously for the Jones polynomial, which is the main subject of Witten's paper. The crux of it is that once everything is formulated in the language of canonical quantization, one can compute partition functions exactly and show that they give rise to different knot polynomials depending on what representation of the gauge group the Wilson loops are in.

Another direction to go in would be to explore the statement of the CS/WZW correspondence [2]. Roughly speaking, the Hilbert spaces appearing in the canonically-quantized CS theory translate to the conformal blocks of WZW theory, which lives on the 2d boundary of the 3-manifold.

# References

[1] E. Witten, *Quantum field theory and the Jones polynomial*, Communications in Mathematical Physics **121**, 351 (1989).

[2] D. C. Cabra and G. L. Rossini, *Explicit connection between conformal field theory and (2+1) Chern-Simons theory*, Mod. Phys. Lett. A **12**, 1687 (1997), hep-th/9506054.

[3] S. Gukov, E. Martinec, G. W. Moore, and A. Strominger, Chern-Simons gauge theory and the AdS(3) / CFT(2) correspondence, in *From Fields to Strings: Circumnavigating Theoretical Physics: A Conference in Tribute to Ian Kogan*, pp. 1606–1647, 2004, hep-th/0403225.

[4] J. Baez and J. P. Muniain, *Gauge fields, knots and gravity* (World Scientific, Singapore, 1994).

[5] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, vol. I* (Wiley and Sons, New York, 1969).

[6] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory*Graduate Texts in Contemporary Physics (Springer-Verlag, New York, 1997).

[7] M. E. Peskin and D. V. Schroeder, *An Introduction to quantum field theory* (Addison-Wesley, Reading, USA, 1995).

# Acknowledgements

I am very grateful to Dr. Sabrina Pasterski for supervising this project. From her I have learned so much new math and physics, and I am constantly amazed by everything she does. This paper would not exist without her generous mentorship and encouragement.

I also thank Professor Ramon van Handel for agreeing to be the 2nd reader for my paper. I thank the staff members of the Program in Applied & Computational Mathematics for being very helpful during my time in the program.

Of course, I also thank my roommate Jesse Lin for being one of the only other physics seniors to survive PACM with me. Our grind sessions were lit.