Prediction and Measurement of WZW Anomaly

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1 Introduction & Roadmap

The low-energy regime of QCD consists of color-singlet bound states: mesons and baryons. One class of mesons consists of the pions, kaons, and η-mesons, which are the eight Goldstone bosons that arise from chiral symmetry breaking. We learned in class that the scattering of pions is to leading order described by the chiral Lagrangian,

$$S_0 = \frac{f_\pi^2}{4} \int \mathrm{d}^4 x \ \mathrm{Tr}\left((\partial_\mu U^\dagger)(\partial^\mu U) \right) . \tag{1.1}$$

eq:chiralL

Here, U is valued in $\mathrm{SU}(N_\pi)$, where $N_\pi = 3$. The pions themselves are defined by

$$U = \exp\left(\frac{2i}{f_\pi} \pi^a T^a \right) , \tag{1.2}$$

where T^a are the $N_\pi^2 - 1$ generators of $\mathrm{SU}(N_\pi)$.

Because the theory of pions is an effective field theory, one can add non-renormalizable higher-derivative correction terms to (1.1). In principle the coefficients should be determined from QCD (but it's hard)[1]. Practically speaking, one can fit the coefficients to experimental data from scattering experiments.

In this paper we focus on one special type of term [3],

$$S_{\mathrm{WZW}} = k \int_B \mathrm{d}^5 x \, \epsilon^{\mu\nu\rho\sigma\tau} \, \mathrm{Tr}\left(U^\dagger(\partial_\mu U) U^\dagger(\partial_\nu U) U^\dagger(\partial_\rho U) U^\dagger(\partial_\sigma U) U^\dagger(\partial_\tau U) \right) . \tag{1.3}$$

eq:wzw

There is a lot of explaining to do here. First, notice that the integral is over a five-dimensional space! We take B to be a five-dimensional ball whose boundary is $\partial B = S^4$ (which we view as the compactification of 4d Euclidean space). Why ever should we consider an action defined over a 5d

[1] Because I'm a bootstrapper, I can't resist pointing out the pion bootstrap papers [1, 2] which, instead of computing directly from QCD, use unitarity and crossing symmetry to constrain the parameter space of coefficients. Notably, [2] places bounds on the chiral anomaly via the pion-photon four-point amplitudes. It seems like they get the correct N_c dependence at least.

space? Well, the physics of this term is only dependent on the boundary of B – in other words, it doesn't actually matter what ball we pick for B as long as the boundary is S^4. This is because the path integral is insensitive to additive $2\pi i$'s in the action, so if $S_{\text{WZW}} = 2\pi i n$ with $n \in \mathbb{Z}$ then all of the dynamics will be the same. Thus, we must have k in (1.3) be an integer.

Given that generic pions carry electric charge, we would like WZW to also be gauged under U(1) symmetry. The question of how to gauge the WZW term is tricky and warrants its own discussion in Section 2. By the end of it we will see that the correct modification of the action is

$$S' = -\,e \int \mathrm{d}^4x A_\mu J^\mu \tag{1.4}$$

eq:sprime

$$+ \frac{ie^2}{24\pi^2} \int \mathrm{d}^4x\, \epsilon^{\mu\nu\rho\sigma}(\partial_\mu A_\nu)A_\rho \,\mathrm{Tr}\left(Q^2(\partial_\sigma U)U^\dagger + Q^2U^\dagger(\partial_\sigma U) + QUQU^\dagger(\partial_\sigma U)U^\dagger\right), \tag{1.5}$$

where Q is the quark charge matrix and J^μ is the U(1) Noether current. Finally the complete chiral action is

$$S_{\text{chiral}} = S_0(\text{with } \partial \mapsto D) + S_{\text{WZW}} + S' \,. \tag{1.6}$$

Importantly, in the five-dimensional term we do *not* elevate ∂ to a covariant derivative; instead S' maintains gauge invariance for us.

What are the possible values of the coefficient k in S_{WZW} and what does it mean physically? We will show in Section **??** that in order to recover the QCD axial anomaly, we must have $k = N_c$, the number of colors in QCD.

Next comes the actual particle physics. Let's think about what interactions we could get from the WZW terms. Schematically, if we just worry about the pions first (without E&M), we can expand

$$U^\dagger\partial_\mu U = \frac{2i}{f_\pi}(\partial_\mu \pi^a)T^a + \mathcal{O}(\pi^2) \tag{1.7}$$

eq:pion

so that the action goes like $(\partial\pi)^5$. Hence we immediately we notice that (1.3) mediates a five-point interaction between pions, which did not exist in the pure chiral theory. The main goal of this paper, however, is to understand the WZW contribution to interactions between pions and photons. Thus it will be necessary to use the full gauged version $S_{\text{WZW}} + S'$. We will find that S' contributes an interaction of the type $A\pi \to \pi\pi$, which, in the vicinity of an atomic nucleus gives rise to $\mathrm{Z}\pi \to \mathrm{Z}\pi\pi$ for some element Z. This is the type of process that can be experimentally measured. Thus, in Section 4 we will compute the amplitude for $AK^+ \to K^+\pi^0$ and discuss the related process Cu $K^+ \to$ Cu $K^+\pi^0$ in Section 5.

The recent OKA collaboration has measured the contribution of the WZW interaction to Cu $K^+ \to$ Cu $K^+\pi^0$. In Section 6 we will discuss the experimental setup of OKA and how they are able to distinguish the contribution from the WZW anomaly.

2 Coupling to photons

sec:photons

Elevating the basic chiral Lagrangian S_0 to a U(1) gauge theory is as simple as sending $\partial_\mu \mapsto D_\mu$, where the covariant derivative acts as

$$D_\mu U = \partial_\mu U + ieA_\mu [Q, U] \ , \tag{2.1}$$

where Q is the charge matrix restricted to the diagonal subgroup,

$$Q = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix} \ . \tag{2.2}$$

What about the WZW term? We may consider doing the same there, but then we would have to extend the covariant derivative to act on the 5d space B, and we certainly do not want any actual physics happening in the bulk space. Let us try something different. We will introduce a new term S' composed of pions coupled to the gauge field which exactly cancels $\delta_{\mathrm{U}(1)} S_{\mathrm{WZW}}$.

Recall that the WZW action

$$S_{\mathrm{WZW}} = k \int_B \mathrm{d}^5 x \, \epsilon^{\mu\nu\rho\sigma\tau} \, \mathrm{Tr} \left(U^\dagger (\partial_\mu U) U^\dagger (\partial_\nu U) U^\dagger (\partial_\rho U) U^\dagger (\partial_\sigma U) U^\dagger (\partial_\tau U) \right) \ . \tag{2.3}$$

eq:wzw1

has a global U(1) symmetry. Let us compute the conserved Noether current. This is an in-principle-trivial but notationally inefficient calculation; to help with that we will use the language of differential forms. For shorthand let us define the 1-form $g = U^\dagger \mathrm{d}U$ so that the WZW term is just

$$S_{\mathrm{WZW}} = k \int_B \mathrm{Tr}(g \wedge g \wedge g \wedge g \wedge g) \ . \tag{2.4}$$

Varying g gives

$$\delta(g \wedge g \wedge g \wedge g \wedge g) = \delta g \wedge g \wedge g \wedge g \wedge g \ . \tag{2.5}$$

eq:gvar

Now we plug in the U(1) variation

$$\delta U(x) = i\varepsilon(x)[Q, U(x)] \tag{2.6}$$

so that

$$\delta g(x) = i\varepsilon(x)[Q, g] + i\, U^\dagger [Q, U] \, \mathrm{d}\varepsilon(x) \tag{2.7}$$

Plugging this in to (2.5), we find that the variation is

$$\delta S_{\mathrm{WZW}} = k \int_B i \, \mathrm{Tr} \left(U^\dagger [Q, U] g \wedge g \wedge g \right) \wedge \mathrm{d}\varepsilon(x) \tag{2.8}$$

$$= k \int_B i \, \mathrm{Tr} \left(\{Q, \mathrm{d}U^\dagger\} \wedge U^\dagger g \wedge g \wedge g \right) \wedge \mathrm{d}\varepsilon(x) \tag{2.9}$$

where we have used the fact that $0 = \mathrm{d}(U^\dagger U)$. Doing a bunch of integration by parts allows us to write this as an exact form,

$$\delta S_{\mathrm{WZW}} = k \int_B i\mathrm{d}\Big[\mathrm{Tr}\left(\{Q, U^\dagger\} U^\dagger g \wedge g \wedge g \right) \wedge \mathrm{d}\varepsilon(x) \Big] \tag{2.10}$$

and finally Stoke's theorem allows us to rewrite the whole thing as an integral over just the boundary,

$$\delta S_{\mathrm{WZW}} = k \int_{\partial B} i\, \mathrm{Tr}\left(\{Q, U^\dagger\} U^\dagger g \wedge g \wedge g \right) \wedge \mathrm{d}\varepsilon(x)\ . \tag{2.11}$$

We identify the term multiplying $\mathrm{d}\varepsilon$ as the 4d U(1) Noether current, and converting back to local coordinates it is

$$J^\mu = i\epsilon^{\mu\nu\rho\sigma}\, \mathrm{Tr}\left(\{Q, U^\dagger\} (\partial_\nu U) U^\dagger (\partial_\rho U) U^\dagger (\partial_\sigma U) \right) . \tag{2.12}$$

Finally, recall that the photon transforms as $\delta A_\mu = \partial_\mu \varepsilon / e$, so let us add the term

$$S''_e = e \int \mathrm{d}^4 x\, A_\mu J^\mu \tag{2.13}$$

to the action to compensate for δS_{WZW}.

We still are not done! $S_{\mathrm{WZW}} + S''_e$ is still not fully gauge invariant because J^μ itself is not gauge invariant to order e, and the entire S'' term goes as e. Computing the next order gives

$$\delta^2 S = i \int_{\partial B} \mathrm{Tr}\left(Q^2 \{U^\dagger, \mathrm{d}U\} + QUQU^\dagger \mathrm{d}U\, U^\dagger \right) . \tag{2.14}$$

Thus we should add another term

$$S''_{e^2} = ie^2 \int_{\partial B} \epsilon^{\mu\nu\rho\sigma} (\partial_\mu A_\nu) A_\rho\, \mathrm{Tr}\left(Q^2 \{U^\dagger, \partial_\sigma U\} + QUQU^\dagger \partial_\sigma U\, U^\dagger \right) \tag{2.15}$$

eq:spp

to the WZW action to make it fully U(1) gauge invariant to order e^2.

3 Matching the axial anomaly

We specialize to $N_c = 3$ from now on. Let us first recall what the fields in the E&M-coupled chiral lagrangian are. The U(1) covariant derivative acts as

$$D_\mu = \partial_\mu + ieQA_\mu\ , \qquad Q = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix} , \tag{3.1}$$

and the full pion octet in SU(3) matrix form is

$$\pi = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta^8}{\sqrt{6}} + \frac{\eta^0}{\sqrt{3}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta^8}{\sqrt{6}} + \frac{\eta^0}{\sqrt{3}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta^8}{\sqrt{6}} + \frac{\eta^0}{\sqrt{3}} \end{pmatrix} . \tag{3.2}$$

Recall from QCD that the triangle diagram contributes to tha anomalous decay of the neutral pion to photons, $\pi^0 \to AA$. Inspecting the first term in (2.15) and using (1.7), we see that it goes like

$$S''_{e^2} \supset k\frac{2i}{f_\pi}\epsilon^{\mu\nu\rho\sigma}(\partial_\mu A_\nu)A_\rho(\partial_\sigma\pi^0) + \mathcal{O}(\pi^2) \tag{3.3}$$

which would mediate $\pi^0 \to AA$, the anomalous triangle diagram from QCD! There are no other ways to add this anomalous term to the chiral Lagrangian. This is one appeal of the WZW term.

Because the QCD triangle diagram is proportional to N_c (the number of quarks that can run in the triangle), we see that in order to recover the correct anomaly we need to set $k = N_c$.

Axial anomaly matching is indirect evidence for the WZW term. What are ways we can probe the direct effects of the presence or absence of the term? It is clear that the WZW term adds many higher-point interactions to the chiral theory. In the remainder of the paper we will focus on one in particular, the $AK^+ \to K^+\pi^0$ process.

4 The $AK^+ \to K^+\pi^0$ amplitude

sec:amp

4.1 WZW term

As promised, let us derive Feynman rules for pions and photons coming from the WZW U(1) gauge term S'. Expanding in the pion field via (1.7), to leading order (1.3) becomes

$$\begin{aligned} S' = \frac{ie}{4\pi^2 f_\pi^3}\epsilon^{\mu\nu\rho\sigma}A_\sigma\Bigg[&\Big((\partial_\mu\pi^+)(\partial_\nu\pi^-) + (\partial_\mu K^+)(\partial_\nu K^-) + (\partial_\mu K^0)(\partial_\nu \bar{K}^0)\Big)\partial_\rho\pi^0 \\ &+ \frac{1}{\sqrt{3}}\Big((\partial_\mu\pi^+)(\partial_\nu\pi^-) + (\partial_\mu K^+)(\partial_\nu K^-) - 3(\partial_\mu K^0)(\partial_\nu \bar{K}^0)\Big)\partial_\rho\eta^8 \\ &+ \sqrt{\frac{2}{3}}\Big((\partial_\mu\pi^+)(\partial_\nu\pi^-) + (\partial_\mu K^+)(\partial_\nu K^-)\Big)\partial_\rho\eta^0\Bigg] . \end{aligned} \tag{4.1}$$

eq:kkpipi

This is a lot of terms, but let us recall the process we care about: $AK^+ \to K^+\pi^0$. Converting to all-in, the interaction is just $AK^+K^-\pi^0$. Thus the term we need is

$$S' \supset \frac{ie}{4\pi^2 f_\pi^3}\epsilon^{\mu\nu\rho\sigma}A_\sigma(\partial_\mu K^+)(\partial_\nu K^-)(\partial_\rho\pi^0) . \tag{4.2}$$

The corresponding Feynman vertex is simply

$$= \frac{e}{4\pi^2 f_\pi^3}\epsilon^{\mu\nu\rho\sigma}p_\mu^{(+)}p_\nu^{(-)}p_\rho^{(0)} \tag{4.3}$$

when all momenta are ingoing. (Outgoing flips $p_\mu \mapsto -p_\mu$). Thus the contribution to the matrix element of the process $AK^+ \to K^+\pi^0$ is simply

$$\mathcal{M}_{\text{WZW}} = -\frac{e}{4\pi^2 f_\pi^3}\epsilon^{\mu\nu\rho\sigma} p_\mu^{(i)} p_\nu^{(f)} p_\rho^{(0)}\, \varepsilon_\sigma(p^{(A)}, \lambda)\,. \tag{4.4}$$

where $p^{(i,f)}$ are the momenta of the initial and final K^+ respectively.

4.2 Tree diagram contributions

In real life, the single-vertex $AK^+K^-\pi^0$ diagram is not the only contribution to the $AK^+ \to K^+\pi^0$ matrix element. Any three-point interaction term of the types $K^+K^-\chi$, $K^+\pi^0\chi$ or $K^+A\chi$ may give rise to tree diagrams that contribute to $AK^+ \to K^+\pi^0$. Here χ could be any meson that interacts with pions and photons. It turns out that the dominant tree-level contributions come from vector meson exchanges, which we would need a further extension of the chiral Lagrangian to describe. Instead of going down that rabbit hole, we will simply use the fact that the contributions have the form [4]

$$\mathcal{M}_{\text{tree}} = \frac{e}{4\pi^2 f_\pi^3}\epsilon^{\mu\nu\rho\sigma} p_\mu^{(+)} p_\nu^{(-)} p_\rho^{(0)}\, \varepsilon_\sigma(p^{(A)}, \lambda)\left(\frac{C_1}{s + i\Gamma_1\sqrt{s} - M_1^2} + \frac{C_2}{t - M_2^2} + \frac{C_3}{u - M_3^2}\right), \tag{4.5}$$

in other words that all three Mandelstam channels appear. Because we will analyze the s-channel only, we allow that channel to have a finite decay width. Thus, this contribution to the matrix element has s-channel poles at $s = M_1^2 - i\Gamma_1$ and $s = -u - M_2^2$ for fixed $u < 0$. In the soft limit $s, t, u \to 0$, these contributions will be suppressed compared to the $\mathcal{M}_{\text{WZW}}$ term.

5 Interaction with Cu nucleus

sec:cu

All we need to know for the purposes of this paper is that, to convert a matrix element for a process involving an external photon and other scalars to a process involving an atomic nucleus and other scalars, we simply take

$$\mathcal{M}(\varepsilon^\mu) \mapsto \mathcal{M}(p_{\text{Z}}^\mu)\frac{\text{Z}}{q^2}F_C(q^2) \tag{5.1}$$

where Z is the nuclear charge, p_{Z}^μ is the momentum of the ingoing nucleus, q is the momentum of the photon exchanged, and F_C is the electromagnetic form factor for the nucleus. In the literature this is called the equivalent photon approximation.

Therefore, importing the results of Section 4, the full matrix element is given by

$$\mathcal{M} = -\frac{e\text{Z}}{4\pi^2 f_\pi^3 q^2}\epsilon^{\mu\nu\rho\sigma} p_\mu^{(i)} p_\nu^{(f)} p_\rho^{(0)} p_\sigma^{(\text{Z})}\left(\lambda_{\text{WZW}} + \frac{C_1}{s + i\Gamma_1\sqrt{s} - M_1^2}\right)F_C(q^2)\,. \tag{5.2}$$

(I've kept only the s-channel term since that's the only one that will be relevant to the experiment later.) Finally, the squared matrix element is

$$|\mathcal{M}|^2 = 24\Big(\frac{eZ}{4\pi^2 f_\pi^3 q^2}\Big)^2 \Big(\lambda_{\text{WZW}} + \frac{C_1^2}{(s-M_1^2)^2+\Gamma_1^2 s}\Big) F_C(q^2) \\ \times \Big((p^{(i)}\cdot p^{(f)})(p^{(0)}\cdot p^{(Z)}) - (p^{(i)}\cdot p^{(0)})(p^{(f)}\cdot p^{(Z)}) + (p^{(i)}\cdot p^{(Z)})(p^{(f)}\cdot p^{(0)})\Big) \tag{5.3}$$

eq:matsquared

This expression can be simplified in terms of five-point Mandelstam invariants. However, the only part that will matter is the $\lambda_{\text{WZW}} + s$-channel pole factor which contains information about the relative sizes of the WZW and tree diagram contributions.

6 The OKA experiment

sec:oka

6.1 Experimental setup

The goal of the OKA experiment [5] is to measure the amplitude of the process Cu $K^+ \to$ Cu $K^+\pi^0$. There are a few ingredients in the setup. First, the experiment uses a K^+ beam ($\sim 8\cdot 10^9$ kaons) directed at a copper target 10 cm in diameter and 2 mm in thickness (~ 1 mole of copper atoms). The outgoing π^0 is detected via its decay into photons, while the outgoing K^+ is indirectly detected by absence of a π^+ signal.

In this setup there could be interference from the decay $K^+ \to \pi^+\pi^0$, which produces a background π^0 signal. The experiment accounts for this by taking out the bands $150 < E < 220$ MeV.

6.2 Matching theory to experiment

The amplitude in question receives contributions not just from WZW but from tree-level interactions as noted in Section 4. Thus, we must be able to distinguish the contributions from those terms. The idea is to scatter close to the $K^*(892)$ (the meson exchanged in the s-channel of $\mathcal{M}_{\text{tree}}$) resonance.[2] At $s \approx m_{K^*}^2$, the denominator of (5.3) can be written approximately as

$$(m_{K^*}^2 - M_1)^2 + \Gamma_1^2 m_{K^*}^2\ . \tag{6.1}$$

Varying the effective $K^+\pi^0$ mass M_1^2 results in a distribution strongly peaked around the mass of the $K^*(892)$. One then fits the parameters m_{K^*} and Γ_1, and λ_{WZW} to match the width and center of the distribution. When the WZW term is allowed, one can obtain values of the mass and decay width of

[2]The analysis in [5] also accounts for contributions from exchange of ω mesons, which contributes a tree diagram factor proportional to $F_S(q^2)$. I decided to neglect it because it doesn't change the gist of the problem.

K^* that are very close to the recorded values in the PDG [6]. On the other hand, when λ_{WZW} is set to zero, taking out the anomaly term, the best-fit values are not within the range of error for the measurements of m_{K^*} and Γ_1. Thus, to correctly predict the properties of the K^* meson, the WZW term is necessary.

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