Wilson loop computation via AdS/CFT

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1 Introduction

In this short essay we review the computation of the quark-antiquark potential in 4d $\mathcal{N}=4$ super-Yang-Mills (SYM) through AdS/CFT. According to the AdS/CFT correspondence, $\mathcal{N}=4$ SYM is dual to type IIB string theory in $\mathrm{AdS}_5 \times S^5$. We use the fact that the Wilson loop on the boundary is dual to the bulk partition function to do the computation in the string theory instead of in the CFT.

$\mathcal{N}=4$ SYM consists of the $\mathrm{SU}(N)$ gauge field A_μ, four fermions λ_a, and six scalars X^I. The action is

$$S = \frac{1}{2g_{\mathrm{YM}}^2} \int \mathrm{d}^4x \, \mathrm{Tr} \left(F_{\mu\nu}F^{\mu\nu} + (D_\mu X^I)^2 + \frac{1}{2}\left[X^I, X^J\right]^2 + 2\bar{\lambda}\not{D}\lambda + \bar{\lambda}\left[\lambda, X\right] \right) . \tag{1.1}$$

This theory is exactly conformal, meaning that $\beta(g_{\mathrm{YM}}) = 0$.

2 $\mathrm{AdS}_5 \times S^5$

Let us briefly review some properties of $\mathrm{AdS}_5 \times S^5$. To start, it is easiest to take $\mathbb{R}^{d,2}$ with signature $(-,+,\ldots,+,-)$ and define the codimension-1 surface

$$-(X^0)^2 + (X^1)^2 + \ldots + (X^d)^2 - (X^{d+1})^2 = -L^2 . \tag{2.1}$$

This is a one-sheeted hyperboloid with radius of curvature L. We define "global AdS" as the universal cover of this surface. Let us parametrize it by

$$\begin{cases} X^0 = \frac{Lt}{z} , \\ X^i = \frac{Lx^i}{z} , \\ X^d = \frac{L^2-(z^2+\vec{x}^2-t^2)}{2z} , \\ X^{d+1} = \frac{L^2+(z^2+\vec{x}^2-t^2)}{2z} . \end{cases} \tag{2.2}$$

for $i = 1, \ldots, d-1$ and $z > 0$. Note that this only covers half of the hyperboloid. In these coordinates the metric is

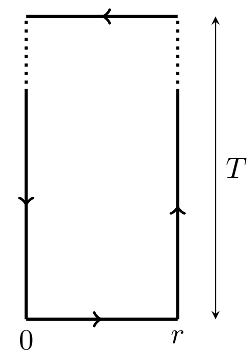
$$ds^2 = \frac{L^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2) = ds^2_{\mathrm{AdS}_5} + L^2 ds^2_{S^5} \ . \tag{2.3}$$

One important feature of AdS that makes it possible to formulate a holographic principle is that it has an asymptotic boundary at infinite proper distance, meaning that null geodesics can actually reach an observer at infinity in finite t.

3 Wilson loop setup

We consider a heavy quark-antiquark pair in the fundamental representation of $\mathrm{SU}(N)$ separated by a distance r, propagating through time $T \to \infty$. Such a system can be described by a Wilson loop,

$$\langle W[C] \rangle = \mathrm{Tr}_{\mathrm{fund}} \left\langle \exp\left(i \oint_C A_\mu \, dx^\mu \right) \right\rangle , \qquad C =$$



. (3.1)

In Euclidean signature this gives

$$\langle W[C] \rangle = e^{-V(r)T} \ , \tag{3.2}$$

and hence we can compute this quantity to read off $V(r)$, the quark-antiquark potential at separation r.

Note that it is important for the probe quarks to be very massive, because otherwise as r increases, it will eventually become more energetically favorable for a new quark-antiquark pair to be created out of vacuum (which is what happens in real-world QCD). In this sense, confinement can only be sharply defined via the Wilson loop in the case of $m_{\mathrm{quark}} \to \infty$. A confining theory in 4d would have

$$V(r) = -\frac{\alpha}{r} + \beta r \ , \tag{3.3}$$

so that the Wilson loop satisfies

$$\langle W[C]\rangle \sim e^{-\mathrm{area}(C)} \ . \tag{3.4}$$

at large r. This is the so-called "area law" for confining gauge theories. On the other hand, if the potential instead has only the Coulomb term, then $V(r) \to$ const. at large r, and so

$$\langle W[C]\rangle \sim e^{-\mathrm{perimeter}(C)} \ . \tag{3.5}$$

Such a theory we call screening, as opposed to confining.

We expect 4d $\mathcal{N} = 4$ SYM to be screening, i.e. $V(r) \sim -1/r$, because it has conformal symmetry and thus no length scale, so quarks scan be separated by arbitrary lengths. Namely, since spacetime dilatation is a symmetry of the theory, we can dilate a quark-antiquark pair to any spacetime separation we want.

4 Dual computation in AdS

As noted above, $\mathcal{N} = 4$ SYM is dual to type IIB string theory in $\mathrm{AdS}_5 \times S^5$. The couplings are related by

$$g_s = \frac{g_{\mathrm{YM}}^2}{4\pi} \ , \quad \frac{L}{\ell_s} = (g_{\mathrm{YM}}^2 N)^{1/4} \ . \tag{4.1}$$

with string length defined by $\ell_s^2 = \alpha'$. For classical gravity (i.e. weak curvature), we should take $N \gg 1$, $g_{\mathrm{YM}}^2 N \gg 1$. This is the usual 't Hooft limit for Yang-Mills theory.

We think of the quark-antiquark pair as living on the boundary of AdS (the "bulk"), and the Wilson loop (3.2) is dual to the string partition function

$$Z = \int_{\partial\Sigma = C} \mathcal{D}X e^{-S[X]} \ , \qquad S = \frac{1}{2\pi\alpha'} \int \mathrm{d}^2\sigma \sqrt{\det \gamma_{ab}} \ , \tag{4.2}$$

where γ_{ab} is the $\mathrm{AdS}_5 \times S^5$ metric (2.3). The meaning of (4.2) is that we should integrate over all surfaces Σ in the bulk which end on the curve C on the boundary. To simplify the calculation, we can make the approximation

$$Z \approx e^{-S_{\mathrm{classical}}[X]} \ , \tag{4.3}$$

where $S_{\mathrm{classical}}$ is a local extremum.

Let us parametrize the worldsheet by $\sigma^1 = t,\ \sigma^2 = x \in [0, r]$. Then z parametrizes the shape of the surface in the bulk, so $z(x)$ is the function we must extremize over. In other words, the worldsheet metric γ_{ab} becomes

$$ds^2 = \frac{L^2}{z^2}(dt^2 + dx^2 + z'^2 dx^2)\ , \tag{4.4}$$

where we have Wick-rotated to Euclidean signature. We have

$$\sqrt{\det \gamma_{ab}} = \frac{\sqrt{1+z'^2}}{z^2} L^2\ , \tag{4.5}$$

so the variational problem is to extremize

$$S[z(x)] = \frac{T}{2\pi\alpha'} L^2 \int_0^r \mathrm{d}x\, \frac{\sqrt{1+z'^2}}{z^2}\ , \tag{4.6}$$

subject to the boundary conditions $z(0) = z(r) = 0$. We can solve this by noting that

$$z^2\sqrt{1+z'^2} = \text{constant}\ , \tag{4.7}$$

and integrating. We find

$$x = \frac{z^3\sqrt{r^4 - z^4}}{3r^4}\, {}_2F_1\big(1, 5/4; 7/4; (z/r)^4\big)\ , \tag{4.8}$$

where ${}_2F_1$ is a hypergeometric function. However, when plugging the answer back in to (4.6) to compute $S_{\text{classical}}$, we find that it diverges, since the asymptotic boundary of AdS has infinite area. Thus, let us impose a cutoff at $z = \varepsilon$ and compute

$$S_{\text{classical}}[z(x)] = \frac{T}{2\pi\alpha'} L^2 \int_\varepsilon^r \mathrm{d}z\, \frac{\sqrt{1+x'^2}}{z^2} = \frac{TL^2}{\pi\alpha'}\left(\frac{1}{\varepsilon} - \frac{4\pi^3}{\Gamma(1/4)^4}\frac{1}{r}\right)\ . \tag{4.9}$$

Ignoring the divergent term, we find that

$$V(r) = -\sqrt{g_{\text{YM}}^2 N}\frac{4\pi^2}{\Gamma(1/4)^4} \times \frac{1}{r}\ , \tag{4.10}$$

which we recognize as the Coulomb potential. This means $\mathcal{N} = 4$ SYM exhibits screening rather than confining behavior.

We can compare this to the weak coupling result which is

$$V(r) \approx -\frac{g_{\text{YM}}^2}{4\pi}\frac{N^2-1}{2N} \times \frac{1}{r}\ . \tag{4.11}$$

Here the N-dependence comes from the value of the quadratic Casimir in the representation that the Wilson loop is in. We can see that although in both cases we recovered the Coulomb potential, the dependence on g_{YM}^2 is quite different.

5 Summary

To summarize, in this essay we computed the Wilson loop in 4d $\mathcal{N} = 4$ SYM using the fact that it is dual to the partition function of a string theory in $\mathrm{AdS} \times S^5$. We found that, at strong coupling, the Wilson loop obeys a perimeter law, indicating that heavy quarks are screened in $\mathcal{N} = 4$ SYM. This qualitatively agrees with the weak coupling result, but the dependence on g_{YM} is different in the two regimes.

References

Silviu Pufu's lectures at PiTP 2023.