S-Matrix Bootstrap for Abelian Scalar EFTs

Preliminary Examination

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11 December 2024

Introduction to S-Matrix Bootstrap

S-matrix bootstrap applies basic physical principles to the (unknown) UV to constrain IR physics.

Two broad motivations:

1. Search for string theory – how special is it? Are there other theories?
2. Experiments can access low-energy physics. What is the minimal data needed to determine the full theory?

Recent bootstrap literature has put bounds on

- Pion scattering (Albert and Rastelli, 2022; Albert et al., 2024a; He and Kruczenski, 2024; Guerrieri et al., 2019)
- Color-ordered scalar scattering in $\mathcal{N} = 4$ SYM (Berman et al., 2024; Berman and Elvang, 2024)
- Scalar scattering in $\mathcal{N} = 8$ SUGRA (Caron-Huot et al., 2021; Albert et al., 2024b)

Two ways to do the bootstrap: *primal* (ruling-in) and $\boxed{\textit{dual}}$ (ruling-out).

We will bootstrap abelian scalars at four-point, with and without SUSY, with and without gravity.

EFT = Effective Field Theory

Consider a massless real scalar in $D = 4$. Write down all possible terms. Schematically:

Wilson coefficients

$$\mathcal{L}_{\text{eff}} = (\partial\phi)^2 + \lambda_{\phi^3}\phi^3 + \sum_{k\geq 0} g_k \, \partial^{2k}\phi^4$$

Renormalizable? Not $\partial^{2k}\phi^4$ for $k > 0$.

(Boring) Candidate UV completion – one massive scalar Φ with $M > M_{\text{gap}}$ that couples to ϕ via an interaction like $\phi\phi\Phi$; no other fields. I.e.

$$\mathcal{L}_{\text{UV}} = \text{kinetic} + \lambda_{\phi^3}\phi^3 + \lambda_{\phi^4}\phi^4 + \frac{1}{2}M^2\Phi^2 + \lambda_{\phi\phi\Phi}\phi\phi\Phi$$

"Integrating out" Φ gives $\mathcal{L}_{\text{eff}}$ with particular values of g_k.

More interesting candidates – abelian open superstring & closed superstring!

Relating UV to IR

Question: Say you give me an EFT with particular Wilson coefficients; how can I tell whether a valid UV completion exists?

"Valid" means

1. unitary,
2. local,
3. analytic in s (except on real axis), &
4. respects symmetries of the problem (e.g. crossing symmetry, SUSY, ...).

Relating UV to IR (cont.)

At all energies, "partial wave expansion"

$$A(s,u) = \sum_j a_j(s)\, P_j(\cos\theta)\ . \tag{1}$$

Unitarity $\implies$ $\mathrm{Im}\, a_j(s) \geq 0 \;\forall\; s, j$ ("positivity"). Call $\mathrm{Im}\, a_j(s) = \rho_j(s)$.

Not manifestly crossing-symmetric.

$t \leftrightarrow u$ symmetry $+$ momentum conservation $\implies$ only even j in the sum.

On the IR side, low-energy expansion

$$A(s,u) = -\lambda_{\phi^3}\left(\frac{1}{s}+\frac{1}{t}+\frac{1}{u}\right)+8\pi G_{\mathrm{N}}\left(\frac{tu}{s}+\frac{us}{t}+\frac{st}{u}\right)+\sum_{m,n\geq 0} g_k^{(m,n)}(s^2+t^2+u^2)^m\,(stu)^n$$

Crossing symmetry $\implies$ manifestly symmetric in $s \leftrightarrow t \leftrightarrow u$. (Other constraints like SUSY are possible, discuss later.)

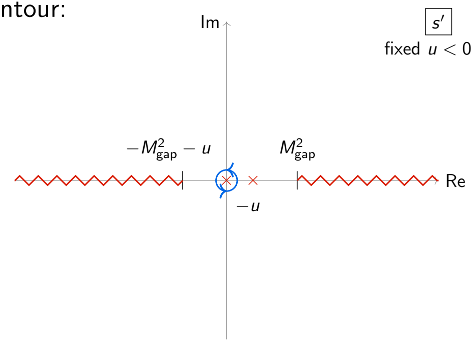
Causality $\implies$ $A(s,u)$ analytic in s $\implies$ **dispersion relations** for g_k.

IR contour

Consider the integral

$$\oint_{s'=0} \frac{\mathrm{d}s'}{2\pi i\, s'} \frac{A(s',u)}{(s')^k}\,, \qquad k \geq n_{\text{sub}}. \tag{2}$$

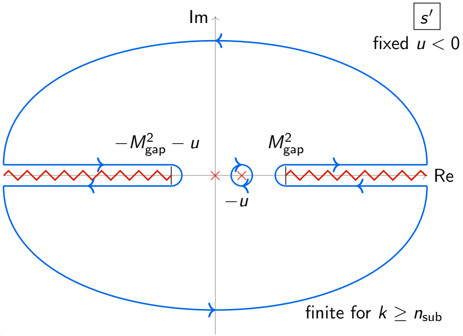
for the following contour:



Assumptions:

1. Weak coupling: no massless loops. *Massive* loops allowed.
2. Mass gap: $\exists$ finite $M_{\text{gap}} > 0$ below which there are no massive states.

UV contour



Assumptions cont'd:

3. Mass cutoff: agnostic about spectrum above some finite $M_{\text{cutoff}} \geq M_{\text{gap}}$.
4. Polynomial-bounded Regge growth ("Froissart bound"):

$$\lim_{\substack{|s|\to\infty \\ \text{fixed } u<0}} \frac{A(s,u)}{s^{n_{\text{sub}}}} = 0 \quad \text{for some } n_{\text{sub}} \in \mathbb{Z}$$

5. For $n_{\text{sub}} = 0$, $\lambda_{\phi^3} = 0$.
For $n_{\text{sub}} = 2$, $G_{\text{Newton}} = 0$.

Dispersion Relations

Dispersion relations have the form

$$g_k^{(m,n)} = \sum_j \int_{M_{\text{gap}}^2}^{\infty} \frac{\mathrm{d}s}{s^{k+1}}\, \rho_j(s)\, w_j^{(m,n)}\,, \quad k \geq n_{\text{sub}}\,. \tag{3}$$

Recall the UV ansatz

$$\operatorname{Im} A(s,u) = \sum_j \rho_j(s)\, P_j(\cos\theta) \tag{4}$$

is not *stu*-symmetric. The resummed amplitude is only *stu*-symmetric if ρ_j's obey

$$0 = \sum_j \int_{M_{\text{gap}}^2}^{\infty} \frac{\mathrm{d}s}{s^{k+1}}\, \rho_j(s)\, c_{k,j}\,, \quad k \geq n_{\text{sub}} \qquad \text{“null constraints”}. \tag{5}$$

Two truncations:

1. truncation in spin: choose some finite set of j's to include in the sum; and
2. truncation in Mandelstam order: choose some $k_{\max}$ up to which we enumerate null constraints.

Only use positivity of ρ, so bound ratios $g_k/g_{n_{\text{sub}}}$.

Basic Analytic Examples

Infinite Spin Tower (IST):

$$A^{\mathrm{IST}}(s,t,u) = \frac{M^6}{(s-M^2)(t-M^2)(u-M^2)} \tag{6}$$

Allowed in 0SDR & 2SDR.

Massive scalar exchange:

$$A^{(0)}(s,t,u) = \frac{M^2}{s-M^2} + \frac{M^2}{t-M^2} + \frac{M^2}{u-M^2} \tag{7}$$

Marginally allowed in 0SDR. Allowed in 2SDR $\implies$ can be subtracted.

$\implies$ **Scalar-subtracted IST (ssIST):**

$$A^{\mathrm{ssIST}}(s,t,u) = A^{\mathrm{IST}}(s,t,u) - |\lambda_0^{(D)}|^2 A^{(0)}(s,t,u) \tag{8}$$

Massive spin-2 exchange:

$$A^{(2)}(s,t,u) = M^2 \left(\frac{P_2^{(D)}(1+\frac{2u}{M^2}) + P_2^{(D)}(1+\frac{2t}{M^2})}{s-M^2} + \mathrm{perms} \right) \tag{9}$$

Not allowed in 0SDR; *marginally allowed* in 2SDR.

Stringy Targets I: Abelian Open Superstring

Tree-level four-point scattering of scalars in Abelian open superstring theory is described by the supersymmetric Dirac-Born-Infeld tree amplitude,

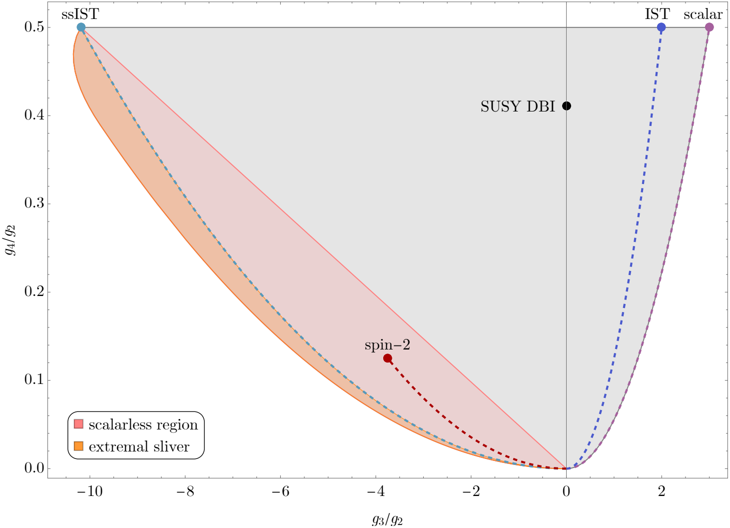
$$A^{\text{DBI}}_{\phi\phi\phi\phi}(s,t,u) = (s^2+t^2+u^2)A^{\text{stripped DBI}}(s,t,u)\ , \quad \text{where} \tag{10}$$

$$A^{\text{stripped DBI}}(s,t,u) = -\alpha'^2\Big(\frac{\Gamma(-\alpha' s)\,\Gamma(-\alpha' u)}{\Gamma(1+\alpha' t)} + \text{perms}\Big) \tag{11}$$

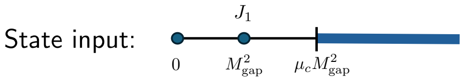
Properties of $A^{\text{stripped DBI}}$:

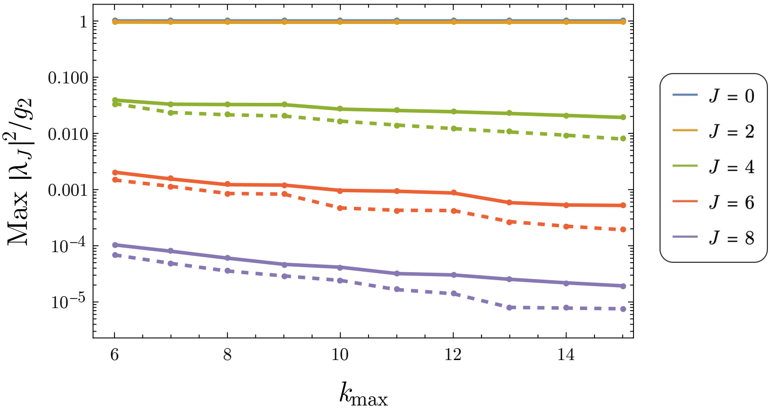
- Satisfies $n_{\text{subs}} = 0$ Froissart bound
- No massless poles
- States appear at $(M/M_{\text{gap}})^2 = 1, 3, 5, \dots$ (setting $\alpha' = 1$) .

Probing the 2SDR Space



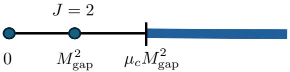
Maximal Spin Constraint

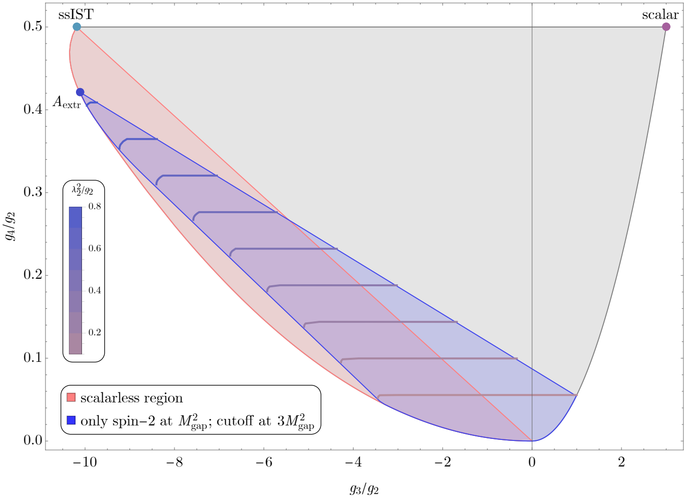




An Extremal Theory

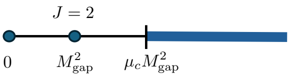
State input:

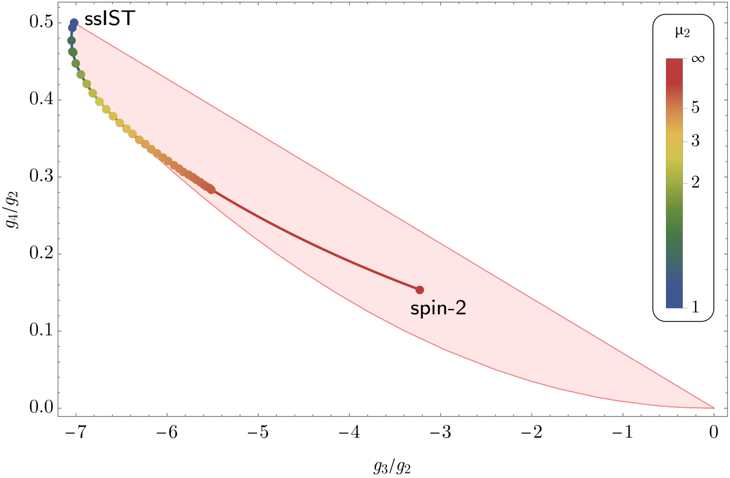




1-Parameter Family of 2SDR Extremal Theories

State input:





Convex Hull Conjecture

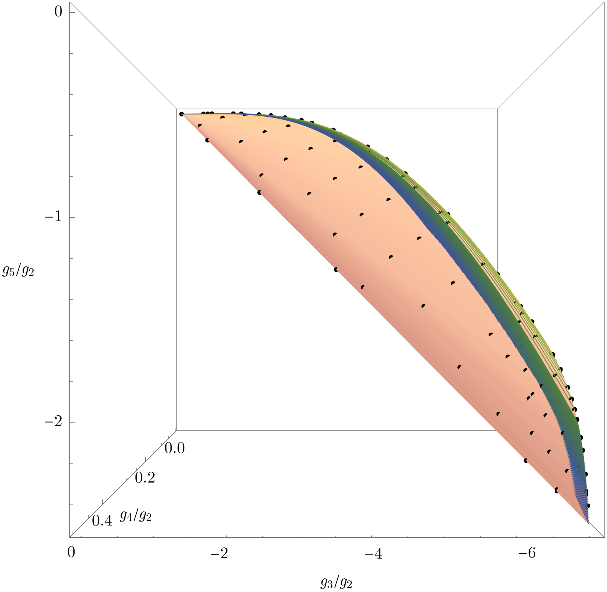
What is required to understand full 2SDR space?

Scalar subtraction allowed $\implies$ full region $=$ scalarless region $+$ scalars.
$\implies$ only need to understand *scalarless* theories!

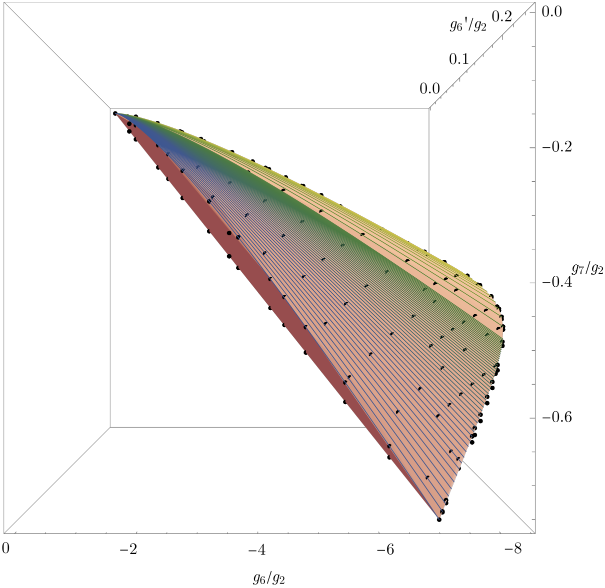
2SDR Convex Hull Conjecture.

The *scalarless* 2SDR allowed space is the convex hull of the 1-parameter family of extremal theories with various choices of mass scale.

Convex Hull Conjecture



Convex Hull Conjecture



Convex Hull Conjecture

Punchline:

If the 2SDR Convex Hull Conjecture is correct, then we only need to understand a single family of theories in order to understand the entire 2SDR space.

Greatly reduces complexity of the problem.

Supersymmetry via 0SDR Bootstrap

In $D = 4$, $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) contains

- 1 vector
- 4 Weyl fermions
- 6 real scalars (3 complex scalars)

Real scalar ϕ is part of a SYM supermultiplet $\implies$ complex scalar $z = \phi + i\varphi$.

From SUSY Ward identities,

$$\begin{aligned} A(zz\bar{z}\bar{z}) &= s^2 \, \frac{A_{\text{photon}}(1^- 2^- 3^+ 4^+)}{\langle 12\rangle^2 [34]^2} \\ &= s^2 \, A^{\text{stripped}}(s, t, u) \, . \end{aligned} \tag{12}$$

To get back to ϕ,

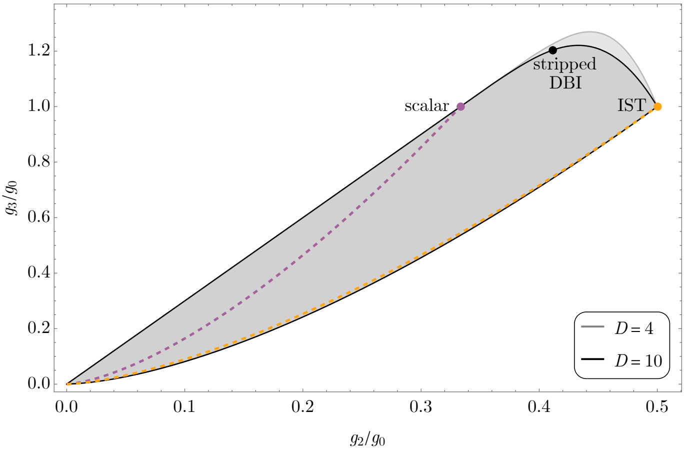
$$\begin{aligned} A(\phi\phi\phi\phi) &= 2A(zz\bar{z}\bar{z}) + 2A(z\bar{z}z\bar{z}) + 2A(\bar{z}zz\bar{z}) \\ &= (s^2 + t^2 + u^2) \, A^{\text{stripped}}(s, t, u) \, . \end{aligned} \tag{13}$$

$A(zz\bar{z}\bar{z})$ has $n_{\text{sub}} = 2 \implies A^{\text{stripped}}$ has $n_{\text{sub}} = 0$.

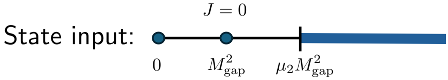
$\implies$ **bootstrap A^{stripped} in 0SDR!**

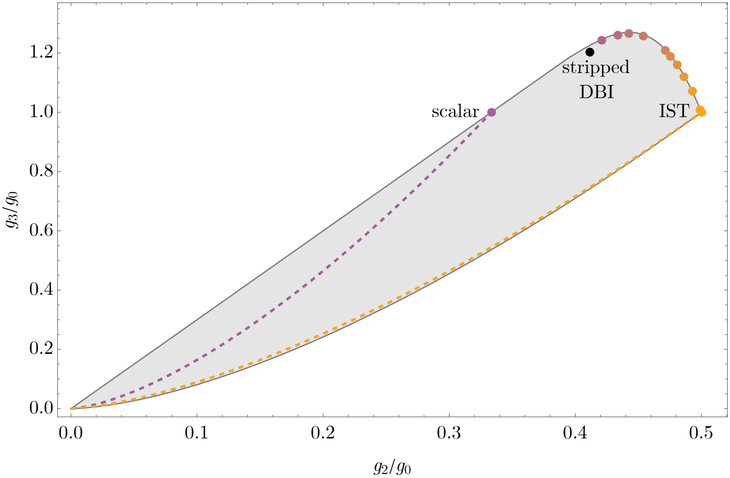
Probing the 0SDR Space

- Scalar is now *marginally* allowed, so cannot be subtracted
- IST is still allowed



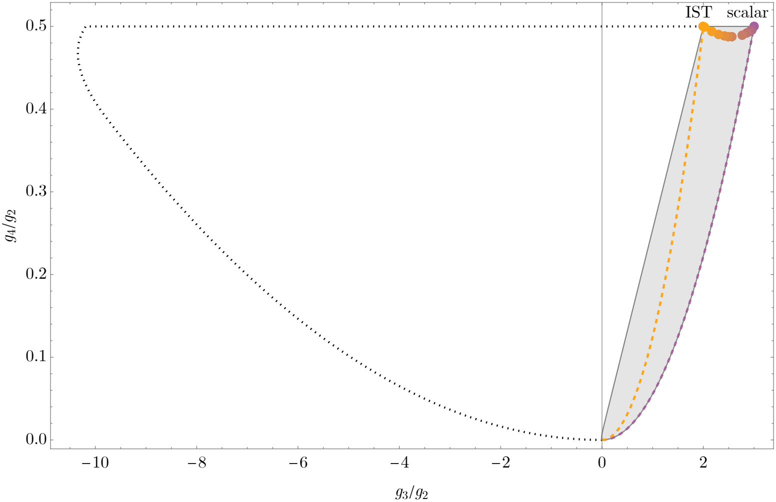
1-Parameter Family of 0SDR Extremal Theories





1-Parameter Family of 0SDR Extremal Theories

To compare w/ 2SDR, normalize by g_2 instead.



0SDR Convex Hull Conjecture

Motivated by SUSY, what is required to understand full 0SDR space?

0SDR Convex Hull Conjecture.

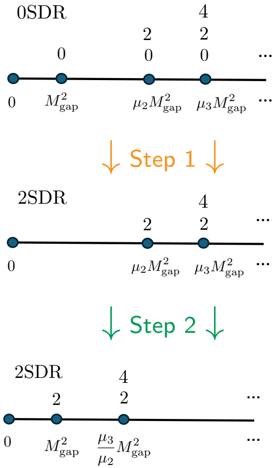
The 0SDR allowed space is the convex hull of the 1-parameter family of extremal theories with various choices of mass scale.

Again, if it's correct, greatly reduces complexity of understanding 0SDR amplitudes.

0SDR $\to$ 2SDR

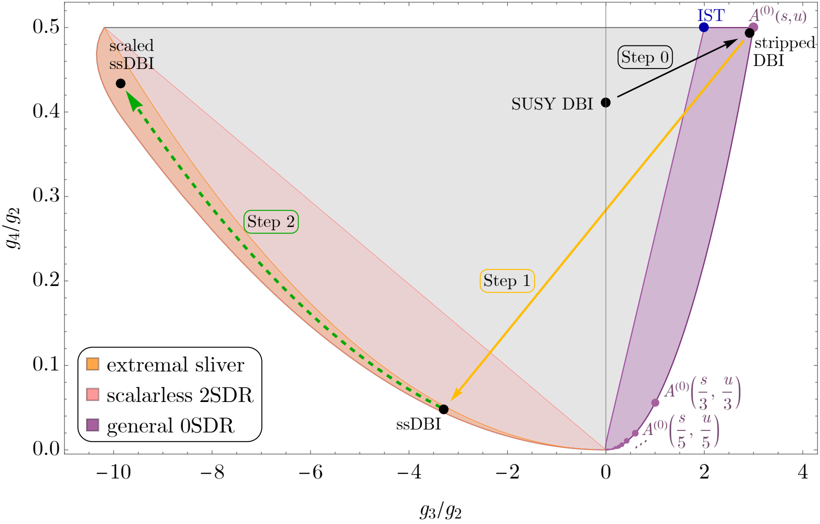
Claim: Understanding 0SDR can actually help us understand 2SDR.

Given any 0SDR theory, can convert to a 2SDR theory via the following procedure:

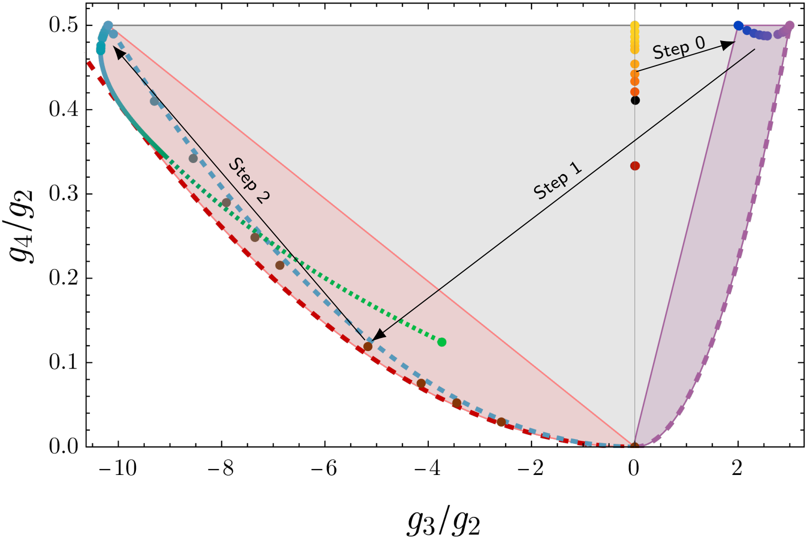


0SDR → 2SDR

E.g. for DBI,



“Everything is SUSY”



Applying the 0SDR→2SDR procedure results in 2SDR amplitudes that appear to lie on the 2SDR scalarless boundary!

"Everything is SUSY"

"Everything is SUSY" Conjecture.

In any D, applying the 0SDR $\to$ 2SDR procedure to the 0SDR extremal theories gives exactly the 2SDR extremal theories.

If true, combine w/ 0SDR Convex Hull Conjecture $\implies$ To understand the general space of abelian scalar amplitudes, one just has to understand a *single family* of SUSY amplitudes.

Limited evidence due to numerical difficulty: can only get to $\mu_2 \approx 1.7$ in 2SDR.

Some analytic candidates being explored by deforming DBI and Virasoro-Shapiro in (Cheung et al., 2024).

Bootstrapping DBI

Recall the SUSY DBI amplitude

$$A^{\text{DBI}}_{\phi\phi\phi\phi}(s,t,u) = -\alpha'^2(s^2+t^2+u^2)\left(\frac{\Gamma(-\alpha's)\,\Gamma(-\alpha'u)}{\Gamma(1+\alpha't)} + \text{perms}\right) \tag{14}$$

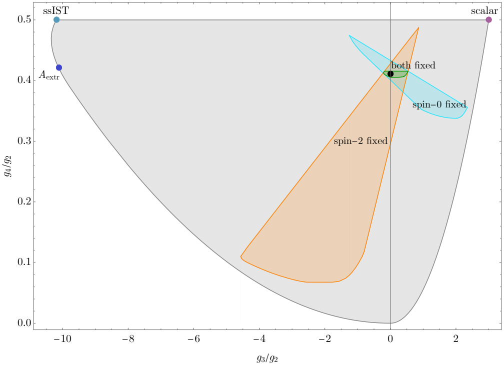
- Satisfies $n_{\text{subs}} = 2$ Froissart bound
- No massless poles
- States at $(M/M_{\text{gap}})^2 = 1, 3, 5, \dots$ (setting $\alpha' = 1$).
  Spins $J = 0, 2$ at gap.

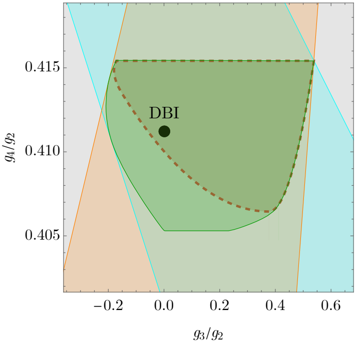
And "stripped DBI",

$$A^{\text{stripped DBI}}(s,t,u) = -\alpha'^2\left(\frac{\Gamma(-\alpha's)\,\Gamma(-\alpha'u)}{\Gamma(1+\alpha't)} + \text{perms}\right) \tag{15}$$

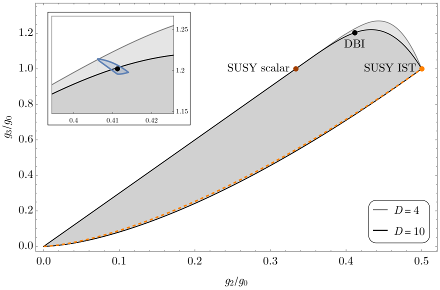
- Satisfies $n_{\text{subs}} = 0$ Froissart bound
- No massless poles
- States at $(M/M_{\text{gap}})^2 = 1, 3, 5, \dots$ (setting $\alpha' = 1$).
  Only spin $J = 0$ at gap.

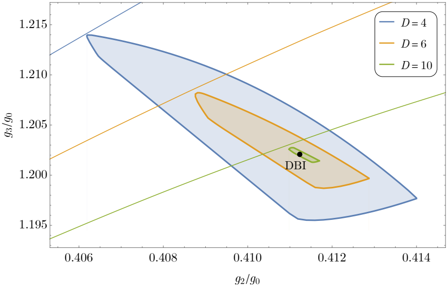
Bootstrapping DBI in 2SDR





Bootstrapping DBI in 0SDR





Stringy Targets II: Closed Superstring

Four-dilaton scattering in Type II closed superstring theory is described by the Virasoro-Shapiro amplitude,

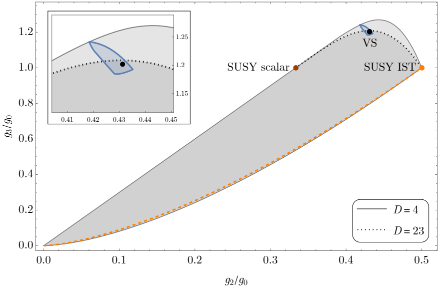
$$A^{\text{dilaton}}_{\phi\phi\phi\phi}(s,t,u) = (s^2+t^2+u^2)^2 A^{\text{VS}}(s,t,u)\ , \quad \text{where} \tag{16}$$

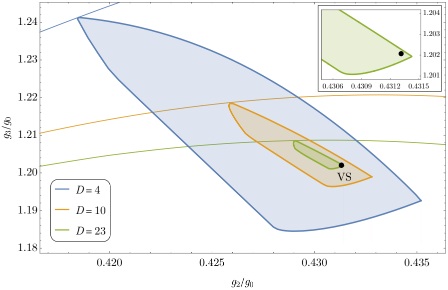
$$A^{\text{VS}}(s,t,u) = -\alpha'^2 \frac{\Gamma(1-\alpha' s)\Gamma(1-\alpha' t)\Gamma(1-\alpha' u)}{\Gamma(1+\alpha' s)\Gamma(1+\alpha' t)\Gamma(1+\alpha' u)} \tag{17}$$

Properties of A^{VS}:

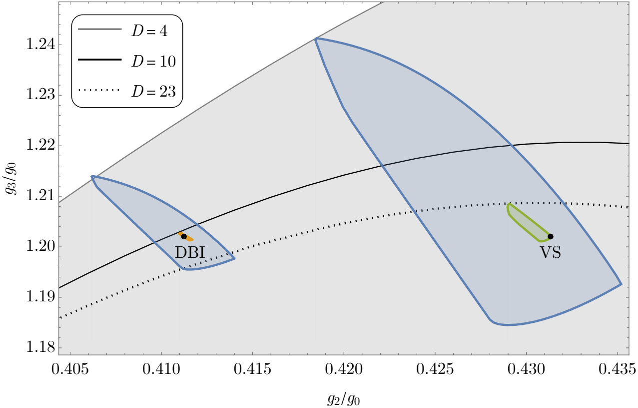
- Satisfies $n_{\text{sub}} = -2$ Froissart bound
- Has massless $1/(stu)$ pole
- States at $(M/M_{\text{gap}})^2 = 1, 2, 3, \ldots$ $(\alpha' = 1)$

Bootstrapping Virasoro-Shapiro in 0SDR





Comparison of 0SDR Stringy Islands



Conditions for Shrinking Islands

1. In order for a tree-level UV completion to be the *unique* theory in its island, it must first be extremal within the general bounds.
2. Analytically continue couplings to general D.

   In order for an amplitude to be extremal in dimension D, it must be borderline nonunitary.

   We call such D the "formal critical dimension" of the amplitude.

Critical dimensions for stringy candidates:

- **DBI:** $D = 10$ for the stripped amplitude $A(\phi\phi\phi\phi)/(s^2 + t^2 + u^2)$ and the full complex scalar $A(zz\bar{z}\bar{z})$ amplitude; $D \approx 10.45$ for the full real scalar amplitude $A(\phi\phi\phi\phi)$.
- **Virasoro-Shapiro:** $D = 23$. (Recall Virasoro-Shapiro already has the SUGRA factor stripped off.)

Note: Don't know if criticality is a *sufficient* condition for extremality.

Summary

Dispersion relations relate UV $\leftrightarrow$ IR. $\implies$ From unitarity of the UV completion, bound low-energy EFT data.

Our work: abelian scalar EFTs. Main results:

1. Extremal theories determined by extremizing coupling to lightest state and specifying mass of next state.
2. Without SUSY, $\{$one-parameter family of scalarless theories$\}$ $\cup$ $\{$scalar-only theories$\}$ determines full space.
3. With SUSY, one-parameter family determines full space.
4. Simple procedure takes SUSY $\rightarrow$ scalarless non-SUSY. Potentially determines entire non-SUSY space!
5. Stringy amplitudes (DBI and Virasoro-Shapiro) best bootstrapped in their formal critical dimension. Criticality seems necessary for extremality.

Next Steps

- How general is the relationship between extremal theories in different subtraction schemes?
  - *su*-symmetric case? (Large-N pions, color-ordered scalars, ...)
  - Other numbers of subtractions?
- Moving away from weak coupling
  - Incorporate loops
  - Implement full unitarity, or at least the upper bound on ρ
  - Compare with primal bounds

Other bootstrap setups to consider:

- Spinning external states (e.g. photons (Bertucci et al., 2024))
- Massive external states
- Mixed systems, e.g. scattering a massive and a massless state

Thank you!

Questions?

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