A Two-Dimensional SU(N) Theory of Quarks

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Junior Paper
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May 7, 2021

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Abstract

$(1 + 1)$-dimensional gauge theories with symmetry group SU(N) are simple models of quantum chromodynamics that can be used to study color confinement. One nonperturbative method for studying these theories is discretized light-cone quantization (DLCQ), in which a null dimension is compactified. Using the DLCQ formalism, we compute the mass spectrum of a $(1 + 1)$-dimensional gauge theory of quarks in the fundamental representation of SU(N). In doing so, we provide strong evidence for the conjectured correspondence between the spectra of this theory and another theory containing an adjoint fermion. This relationship can shed light on the surprising fact that the adjoint theory exhibits screening, as opposed to confining, behavior in the massless case. We use the lowest energy states in the spectrum of our theory to compute the height of the interaction potential between massive probe quarks in the adjoint theory. Additionally, we use first-order perturbation theory to study the mass dependence in the low mass limit. We generalize these analyses to theories with any integer ratio of flavors to colors, which are related to adjoint theories with the corresponding number of adjoint flavors.

This paper represents my work in accordance with University regulations.
/s/ Loki Lin 

Contents

1 Introduction

The phenomenon of color confinement, which states that free quarks do not appear in nature, can be observed in experiments and in numerical simulations of quantum chromodynamics (QCD), but has yet to be derived analytically from first principles. In particular, [1] has proposed that the interaction between quarks and antiquarks can be approximated by the Cornell potential $V(r) = -\kappa/r + \beta r$, where r is the separation between the quark-antiquark pair. The parameters in $V(r)$ can be determined by computing the meson spectrum of the theory and comparing with experimental data, as detailed in [2, 3]. Figure 1 plots this potential with experimentally-determined values for the constants κ, β from [3]. One can see that the potential grows linearly at large separations. However, the large r regime cannot be observed in the real world because, in four-dimensional QCD with dynamical quarks, a new quark-antiquark pair is created whenever the separation gets too large. In addition, QCD cannot be studied via perturbative methods due to strong coupling. Due to these difficulties, it is useful to study two-dimensional "toy" models using nonperturbative methods.

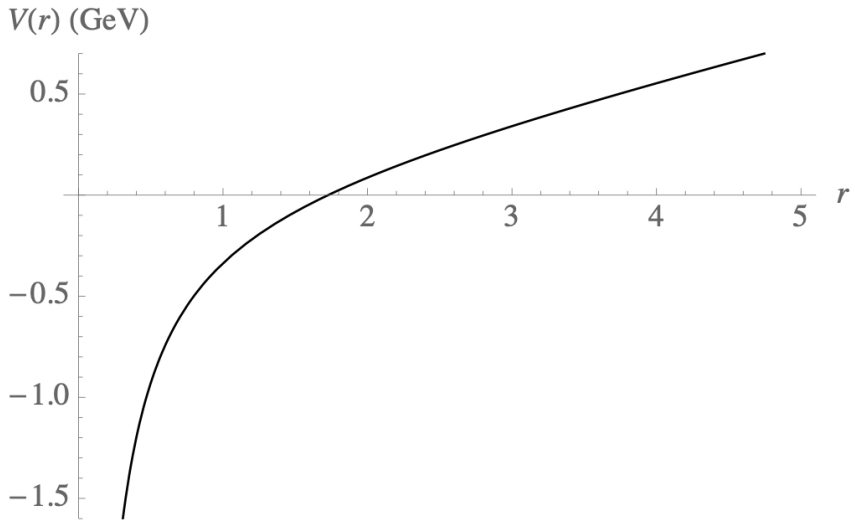


Figure 1: The Cornell potential $V(r) = -\kappa/r + \beta r$ with parameters $\kappa = 0.507$, $\beta = 0.17$ GeV^2 from [3].

One can construct a model of QCD by using the formalism of gauge theory, where the structure group describes the gauge symmetry of the theory. Real-world QCD has symmetry group SU(3); we can generalize this to SU(N). By considering different representations of SU(N), we can obtain different types of fermions. Two such types, the fundamental and the adjoint, are commonly studied. Fermions in the fundamental representation are referred to as quarks; adjoint fermions are Majorana, i.e. real, whereas quarks are Dirac, i.e. complex. We can also add flavors of each type of fermion with different masses. Another useful tool is to consider the $N \to \infty$ limit, which simplifies calculations as in [4]. Moreover, [5] shows that the large N limit suppresses pair creation for dynamical fundamental quarks, so the interaction potential can be studied at any separation.

QCD theories in two dimensions have unique and interesting behaviors. For example, the theory with an adjoint quark and no dynamical fundamental quarks, studied in [6, 7], exhibits screening behavior in the massless case, i.e. non-dynamical "probe" quarks be-

come free instead of being confined. This is unexpected because screening usually arises from the creation of fundamental quark-antiquark pairs, and in the adjoint theory there are no fundamental quarks to do the screening. To be more precise: probe quarks in the theory form mesonic states, and one can view such a state as a quantum mechanical system governed by a relativistic Schrodinger equation with some interaction potential between the particles. [6, 7] predict that this potential is linear at small separations and levels off to some finite limit at infinity, as displayed qualitatively in Figure 2. Such a potential must have scattering states, which is why screening occurs.

Figure 2: A qualitative plot of the quark-antiquark interaction potential for probe quarks in the theory with one massless adjoint.

It was proposed in [7, 8] that the spectrum of this theory is related to that of another SU(N) gauge theory whose matter content consists of N_f flavors of quarks in the fundamental representation, with one quark massive and the others massless. Namely, one can construct a particular Lie algebra called a Kac-Moody (KM) algebra from the Fourier modes of the SU(N) gauge current of the theory. KM algebras are classified by their KM level, a parameter that appears in the commutation relations for the algebra. In other words, for theories with different matter content, the construction of the SU(N) gauge current in terms of the oscillator modes in each theory will be different, but the commutation relations for the current algebras will look the same, up to KM level. The Hilbert space of the theory is then generated by acting with Fourier modes of the current on a primary state which transforms in some representation of SU(N).

It turns out that the Hamiltonian for such a theory can be written just in terms of Fourier modes of the gauge current, so its spectrum restricted to a subspace generated by some primary state depends solely on the KM level of the current algebra and the SU(N) representation of the primary. Thus, if two theories have the same KM level, their spectra in subspaces generated by KM primaries in a particular representation of SU(N) will be the same. [7] constructs KM primaries for the adjoint theory in the case where the probe quarks are massless. In the massive case, however, the primaries are difficult to construct, so it is useful to instead check these facts by explicitly computing the spectra of these theories. This is the main motivation for our paper. Additionally, the theory with fundamental quarks can be used to find the energy at which the spectrum of the adjoint theory becomes continuous. This corresponds to the height of the potential between probe quark-antiquark pairs in mesonic states of the adjoint theory.

In fact, a similar correspondence exists for theories with any number of adjoints. Such multi-adjoint theories are related to generalized versions of our theory where the number of massless flavors, $N_f - 1$, is an integer multiple of the number of colors, N. The multi-adjoint theories are also screening when the adjoints are massless, so we can use our generalized theory to compute the height of the interaction potential for this family of theories.

We employ a method called discretized light-cone quantization (DLCQ) to study our model numerically, as in [7, 9, 10]. DLCQ is a nonperturbative method which involves working in null coordinates and compactifying one of the dimensions so that the momentum in that direction takes on discrete values. We then get finitely many states at any fixed resolution, so the problem is reduced to finding the spectra of finite-dimensional matrices. [7] uses the adjoint theory to reconstruct the spectrum of our theory. We find that the spectrum matches this prediction, which verifies the correspondence between the theories.

This paper begins by formulating the theory and deriving the operators of interest in Section 2. We then detail our computations and numerical results in Section 3.

We work in natural units with $\hbar = c = 1$.

2 Relevant background

Schematically, the procedure for constructing the theory is as follows. We define an SU(N) gauge theory and its action; we then quantize the fields in null coordinates and compactify one dimension. Finally, we associate to the theory a Hilbert space of states, which are the mesons we are interested in.

2.1 Setup

Spacetime is $\mathcal{M} = \mathbb{R}^{1+1}$ with the Minkowski metric $\mathrm{d}s^2 = (\mathrm{d}x^0)^2 - (\mathrm{d}x^1)^2$. Our representation of the Clifford algebra is $\gamma^0 = \sigma_2, \gamma^1 = i\sigma_1$. One can check that these satisfy the Clifford algebra relation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

We transform to null coordinates, also known as light-cone coordinates. These are related to the usual (x^0, x^1) Minkowski coordinates by $x^\pm = (x^0 \pm x^1)/\sqrt{2}$. In these coordinates the metric becomes $\mathrm{d}s^2 = 2\mathrm{d}x^+\,\mathrm{d}x^-$. For the purposes of the field theory later on, we will consider x^+ as "time" and x^- as "space", even though they are not actually timelike or spacelike in the geometric sense. This prescription is simply a matter of convention.

The theory we would like to study consists of N_f fermions $q_{i\alpha}$ in the fundamental representation of the Lie group SU(N); some facts about SU(N) and its representation theory can be found in Appendix A.1. We use Latin letters to denote SU(N) indices and Greek letters to denote flavor indices (with the exception of μ, ν, which will be used to denote spacetime coordinate indices), i.e. $\alpha \in \{1, \ldots, N_f\}$ and $i \in \{1, \ldots, N\}$. The first $N_f - 1$ fermions are massless and the last fermion has mass m_{fund}, which we will often abbreviate to m when there is no ambiguity.

More concretely, we have a topologically trivial principal SU(N) bundle, and our fermions are sections of fibers of the associated vector bundle consisting of the direct

product of N_f copies of the fundamental representation. The gauge field is a traceless Hermitian connection $(A_\mu)_{ij}$ on the bundle. The covariant derivative is given by $D_\mu q_{i\alpha} = \partial_\mu q_{i\alpha} + i(A_\mu)_{ij} q_{j\alpha}$, which differs from the mathematicians' convention by a factor of i. The details of this geometric formalism can be found in Appendix A.2.

We also note that, in addition to the local $\mathrm{SU}(N)$ symmetry, we have a global $\mathrm{SU}(N_f - 1)$ action on the subspace of each fiber corresponding to the massless fermions, considering it as a $(N_f - 1)$-dimensional vector space. It turns out that the states we will eventually construct will be in some representation of $\mathrm{SU}(N_f - 1)$ under this action, so we can use such representations to classify states.

In what follows, it will be useful to split the fields into components: $A_\pm = (A_0 \pm A_1)/\sqrt{2}$, $q^\mu_{i\alpha} = 2^{-1/4} \begin{pmatrix} v_{i\alpha} \\ \phi_{i\alpha} \end{pmatrix}$. Let us fix the gauge $A_- = 0$.

The following subsections, especially Sections 2.2 through 2.5, reference the construction of the theories in [4,7,9,10]. We also reference [11] for some facts about DLCQ. All of our conventions follow [7].

2.2 Action

The action is given by

$$S = \int \mathrm{d}^2x \left[\mathrm{Tr}\left(\frac{-1}{4g^2} F_{\mu\nu} F^{\mu\nu} \right) + i \bar{q}_\alpha \not{D} q_\alpha - m_\alpha \bar{q}_\alpha q_\alpha \right] \tag{2.1}$$

where $m_\alpha = 0$ for $\alpha = 1, \ldots, N_f - 1$, $m_{N_f} = m_{\text{fund}}$. $F_{\mu\nu}$ is the curvature 2-form given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$. This action is manifestly invariant under the $\mathrm{SU}(N)$ structure group, the $\mathrm{SU}(N_f - 1)$ global symmetry, and the group of continuous isometries of $\mathcal{M}$.

It is useful to calculate each term in (2.1) explicitly. First we have

$$F_{\mu\nu} F^{\mu\nu} = -2(\partial_+ A_- - \partial_- A_+ + i[A_+, A_-])^2 = -2(\partial_- A_+)^2 \tag{2.2}$$

since we chose the gauge $A_- = 0$. We also have

$$\bar{q}_\alpha \not{D} q_\alpha = q^\dagger_{\alpha i} \gamma^0 \gamma^\mu (\partial_\mu q_{i\alpha} + i(A_\mu)_{ij} q_{j\alpha}) = \phi^\dagger_{\alpha i} \partial_- \phi_{i\alpha} + v^\dagger_{\alpha i} \partial_+ v_{i\alpha} - i v^\dagger_{\alpha i} (A_+)_{ij} v_{j\alpha} \tag{2.3}$$

and

$$\bar{q}_\alpha q_\alpha = q^\dagger_\alpha \gamma^0 q_\alpha = \frac{i}{\sqrt{2}} (\phi^\dagger_{\alpha i} v_{i\alpha} - v^\dagger_{\alpha i} \phi_{i\alpha}) \tag{2.4}$$

where we have switched to null coordinates for the fields in the second steps. Putting these together gives us

$$S = \int \mathrm{d}^2x \left[\mathrm{Tr}\left(\frac{1}{2g^2} (\partial_- A_+)^2 \right) + i \phi^\dagger_{\alpha i} \partial_- \phi_{i\alpha} + i v^\dagger_{\alpha i} \partial_+ v_{i\alpha} - v^\dagger_{\alpha i} (A_+)_{ij} v_{j\alpha} - \frac{i m_\alpha}{\sqrt{2}} (\phi^\dagger_{\alpha i} v_{i\alpha} - v^\dagger_{\alpha i} \phi_{i\alpha}) \right] \tag{2.5}$$

We have a conserved current associated with the $\mathrm{SU}(N)$ gauge symmetry, which we can read off from (2.5) as

$$(J^+)_{ij} = -v^\dagger_{\alpha j} v_{i\alpha} + \frac{1}{N} \delta_{ij} v^\dagger_{\alpha k} v_{k\alpha} \tag{2.6}$$

where we have subtracted out the trace. Then we have the identity

$$\mathrm{Tr}(A_+J^+) = (A_+)_{ji}(J^+)_{ij} = (A_+)_{ji}v_{i\alpha}v^\dagger_{\alpha j} - \frac{1}{N}(\mathrm{Tr}A_+)v^\dagger_{\alpha k}v_{k\alpha} = -(A_+)_{ji}v^\dagger_{\alpha j}v_{i\alpha} \tag{2.7}$$

since $\mathrm{Tr}A_+ = 0$, so we can rewrite (2.5) as

$$S = \int \mathrm{d}^2x \big[\mathrm{Tr}\big(\frac{1}{2g^2}(\partial_-A_+)^2 + A_+J^+\big) + i\phi^\dagger_{\alpha i}\partial_-\phi_{i\alpha} + iv^\dagger_{\alpha i}\partial_+v_{i\alpha} - \frac{im_\alpha}{\sqrt{2}}(\phi^\dagger_{\alpha i}v_{i\alpha} - v^\dagger_{\alpha i}\phi_{i\alpha})\big] \tag{2.8}$$

As explained in Section 2.1, we consider x^+ as the time coordinate and x^- as the space coordinate. Following this prescription, one can see that A_+ and $\phi_{i\alpha}, \phi^\dagger_{\alpha i}$ are non-dynamical in the sense that their derivatives with respect to the time coordinate x^+ do not appear in the action (2.8). We can use the equations of motion to eliminate them:

$$\frac{1}{g^2}\partial_-^2A_+ = J^+ \tag{2.9}$$

$$\partial_-\phi^\dagger_{\alpha i} = \frac{1}{\sqrt{2}}m_\alpha v^\dagger_{\alpha i},\ \partial_-\phi_{i\alpha} = \frac{1}{\sqrt{2}}m_\alpha v_{i\alpha} \tag{2.10}$$

One can define the "inverse partial derivative" which is the inverse of the derivative operator as defined in Fourier space (i.e. the operator which is dual to division in Fourier space). Namely, let $\mathcal{F}$ denote the Fourier transform operator:

$$\mathcal{F}[f](k) = \frac{1}{\sqrt{2\pi}}\int \mathrm{d}x\, f(x)e^{-ikx} \tag{2.11}$$

We have the identity $\mathcal{F}[\partial f](k) = ik\mathcal{F}[f](k)$ from integration by parts, so it makes sense to define $1/\partial$ as satisfying $\mathcal{F}[\frac{1}{\partial}f](k) = -\frac{i}{k}\mathcal{F}[f](k)$. In terms of $1/\partial$, we have

$$\int \mathrm{d}^2x\,(\partial_-A_+)^2 = -\int \mathrm{d}^2x\,A_+\partial_-^2A_+ = -\int \mathrm{d}^2x\,g^4J^+\frac{1}{\partial_-^2}J^+ \tag{2.12}$$

where in the first step we have integrated by parts. The other two equations are

$$\phi_{i\alpha} = \frac{m_\alpha}{\sqrt{2}}\frac{1}{\partial_-}v_{i\alpha},\ \phi^\dagger_{\alpha i} = \frac{m_\alpha}{\sqrt{2}}\frac{1}{\partial_-}v^\dagger_{\alpha i} \tag{2.13}$$

Using the above to rewrite (2.8), we arrive at the final form of the action:

$$S = \int \mathrm{d}^2x \big[\mathrm{Tr}\big(\frac{g^2}{2}J^+\frac{1}{\partial_-^2}J^+\big) + iv^\dagger_{\alpha i}\partial_+v_{i\alpha} + \frac{im_\alpha^2}{2}v^\dagger_{\alpha i}\frac{1}{\partial_-}v_{i\alpha}\big] \tag{2.14}$$

2.3 Momentum operators

We can write down the Hamiltonian density, which we will call $\mathcal{P}^-$:

$$\mathcal{P}^- = \frac{\partial\mathcal{L}}{\partial(\partial_-v_{i\alpha})}\partial_-v_{i\alpha} + \frac{\partial\mathcal{L}}{\partial(\partial_-\phi_{i\alpha})}\partial_-\phi_{i\alpha} + \mathrm{h.c.} - \mathcal{L} \tag{2.15}$$

$$= -\mathrm{Tr}\big(\frac{1}{2g^2}(\partial_-A_+)^2 + A_+J^+\big) - iv^\dagger_{\alpha i}\partial_+v_{i\alpha} + \frac{im_\alpha}{\sqrt{2}}(\phi^\dagger_{\alpha i}v_{i\alpha} - v^\dagger_{\alpha i}\phi_{i\alpha}) \tag{2.16}$$

The same simplification as before yields

$$\mathcal{P}^- = -\mathrm{Tr}\big(\frac{g^2}{2} J^+ \frac{1}{\partial_-^2} J^+\big) - \frac{i m_\alpha^2}{2} v_{\alpha i}^\dagger \frac{1}{\partial_-} v_{i\alpha} \tag{2.17}$$

Integrating this over the spatial variable gives the Hamiltonian operator

$$P^- = \int \mathrm{d}x^- \big[-\mathrm{Tr}\big(\frac{g^2}{2} J^+ \frac{1}{\partial_-^2} J^+\big) - \frac{i m_\alpha^2}{2} v_{\alpha i}^\dagger \frac{1}{\partial_-} v_{i\alpha}\big] \tag{2.18}$$

The spatial momentum density $\mathcal{P}^+$ is the $(+,-)$ component of the mixed stress-energy tensor $T^\mu{}_\nu$, which is given similarly by

$$\mathcal{P}^+ = T^+{}_- = \frac{\partial \mathcal{L}}{\partial(\partial_+ v_{i\alpha})} \partial_- v_{i\alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_+ \phi_{i\alpha})} \partial_- \phi_{i\alpha} + \text{h.c.} = i v_{\alpha i}^\dagger \partial_- v_{i\alpha} \tag{2.19}$$

Thus we have the spatial momentum operator

$$P^+ = \int \mathrm{d}x^- i v_{\alpha i}^\dagger \partial_- v_{i\alpha} \tag{2.20}$$

2.4 Quantization

The procedure for quantization is to first expand the fields in Fourier space, which makes momentum conservation explicit, and then to quantize each mode with canonical anti-commutation relations, thus defining creation and annihilation operators. This scheme is reminiscent of the quantization of the simple harmonic oscillator from nonrelativistic quantum mechanics; each of our modes can be thought of as a fermionic oscillator.

Following this, we begin with the expansions of $v_{i\alpha}, v_{\alpha i}^\dagger$ in Fourier modes:

$$v_{i\alpha}(x^-) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{d}k^+ (d_{i\alpha} e^{-ik^+x^-} + c_{\alpha i}^\dagger e^{ik^+x^-}) \tag{2.21}$$

$$v_{\alpha i}^\dagger(x^-) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{d}k^+ (d_{\alpha i}^\dagger e^{ik^+x^-} + c_{i\alpha} e^{-ik^+x^-}) \tag{2.22}$$

then impose the equal-time anticommutation relations

$$\big\{d_{i\alpha}(k_1^+), d_{\beta j}^\dagger(k_2^+)\big\} = \big\{c_{i\alpha}(k_1^+), c_{\beta j}^\dagger(k_2^+)\big\} = \delta_{ij}\delta_{\alpha\beta}\,\delta(k_1^+ - k_2^+) \tag{2.23}$$

with all other combinations being 0. The anticommutators of the fields are then

$$\big\{v_{i\alpha}(x_1^-), v_{\beta j}^\dagger(x_2^-)\big\} = \frac{1}{2\pi} \int_{\mathbb{R}_+^2} \mathrm{d}^2k^+ \Big(\big\{d_{i\alpha} e^{-ik_1^+x_1^-}, d_{\beta j}^\dagger e^{ik_2^+x_2^-}\big\} + \big\{c_{\alpha i}^\dagger e^{ik_1^+x_1^-}, c_{j\beta} e^{-ik_2^+x_2^-}\big\}\Big) \tag{2.24}$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathrm{d}^2k^+ \,\delta_{ij}\delta_{\alpha\beta}\,\delta(k_1^+ - k_2^+)\, e^{i(k_1^+x_1^- - k_2^+x_2^-)} = \delta_{ij}\delta_{\alpha\beta}\,\delta(x_1^- - x_2^-) \tag{2.25}$$

where in the last step we have integrated out the exponential. The other anticommutators vanish:

$$\big\{v_{i\alpha}(x_1^-), v_{j\beta}(x_2^-)\big\} = \big\{v_{\alpha i}^\dagger(x_1^-), v_{\beta j}^\dagger(x_2^-)\big\} = 0 \tag{2.26}$$

These are exactly the desired equal-time anticommutation relations for fermionic fields. We interpret $d^\dagger(k), c^\dagger(k)$ as creation operators and $d(k), c(k)$ as annihilation operators

for the kth mode.

Let us compute the momenta in terms of these modes. P^+ is straightforward. First we compute the momentum density:

$$\mathcal{P}^+ = \frac{i}{2\pi}\int_{\mathbb{R}^2_+} \mathrm{d}^2k (d^\dagger_{\alpha i}e^{ik_1x} + c_{i\alpha}e^{-ik_1x})\partial_x(d_{i\alpha}e^{-ik_2x} + c^\dagger_{\alpha i}e^{ik_2x}) \tag{2.27}$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}^2_+} \mathrm{d}^2k\, k_2\big(d^\dagger_{\alpha i}d_{i\alpha}e^{i(k_1-k_2)x} - d^\dagger_{\alpha i}c^\dagger_{\alpha i}e^{i(k_1+k_2)x} + d_{i\alpha}c_{i\alpha}e^{-i(k_1+k_2)x} - c_{i\alpha}c^\dagger_{\alpha i}e^{-i(k_1-k_2)x}\big) \tag{2.28}$$

Thus the momentum operator is

$$\begin{aligned} P^+ = \frac{1}{2\pi}\int_{\mathbb{R}^2_+} \mathrm{d}^2k\, k_2\big(&d^\dagger_{\alpha i}d_{i\alpha}\delta(k_1-k_2) - d^\dagger_{\alpha i}c^\dagger_{\alpha i}\delta(k_1+k_2) \\ &+d_{i\alpha}c_{i\alpha}\delta(k_1+k_2) - c_{i\alpha}c^\dagger_{\alpha i}\delta(k_1-k_2)\big) \end{aligned} \tag{2.29}$$

We can simplify this by integrating over k_1 or k_2. Either way we get

$$P^+ = \int_0^\infty \mathrm{d}k\, k(d^\dagger_{\alpha i}d_{i\alpha} - c_{i\alpha}c^\dagger_{\alpha i}) \tag{2.30}$$

To compute P^-, we split it into two parts, P^-_{mass} and P^-_{int}. The computation for P^-_{mass} is nearly identical to the one for P^+ and gives

$$P^-_{\text{mass}} = \frac{m^2}{2}\int_0^\infty \mathrm{d}k\, \frac{1}{k}(d^\dagger_{\alpha i}d_{i\alpha} - c_{i\alpha}c^\dagger_{\alpha i}) \tag{2.31}$$

A more arduous computation for P^-_{int} gives

$$\begin{aligned} P^-_{\text{int}} = \frac{g^2}{4\pi}\int_{\mathbb{R}^4_+} \mathrm{d}^4k\, \Big(&\frac{d_{i\alpha}d^\dagger_{\alpha j}d_{j\beta}d^\dagger_{\beta i}}{(k_3-k_4)^2}\delta(k_3-k_4+k_1-k_2) + \frac{d_{i\alpha}d^\dagger_{\alpha j}d_{j\beta}c_{i\beta}}{(k_3+k_4)^2}\delta(k_3+k_4+k_1-k_2) \\ &+ \frac{d_{i\alpha}d^\dagger_{\alpha j}c^\dagger_{\beta j}d^\dagger_{\beta i}}{(k_3+k_4)^2}\delta(k_3+k_4-k_1+k_2) + \frac{d_{i\alpha}d^\dagger_{\alpha j}c^\dagger_{\beta j}c_{i\beta}}{(k_3-k_4)^2}\delta(k_3-k_4-k_1+k_2) \\ &+ \frac{d_{i\alpha}c_{j\alpha}d_{j\beta}d^\dagger_{\beta i}}{(k_3-k_4)^2}\delta(k_3-k_4+k_1+k_2) + \frac{d_{i\alpha}c_{j\alpha}d_{j\beta}c_{i\beta}}{(k_3+k_4)^2}\delta(k_3+k_4+k_1+k_2) \\ &+ \frac{d_{i\alpha}c_{j\alpha}c^\dagger_{\beta j}d^\dagger_{\beta i}}{(k_3+k_4)^2}\delta(k_3+k_4-k_1-k_2) + \frac{d_{i\alpha}c_{j\alpha}c^\dagger_{\beta j}c_{i\beta}}{(k_3-k_4)^2}\delta(k_3-k_4-k_1-k_2) \\ &+ \frac{c^\dagger_{\alpha i}d^\dagger_{\alpha j}d_{j\beta}d^\dagger_{\beta i}}{(k_3-k_4)^2}\delta(k_3-k_4-k_1-k_2) + \frac{c^\dagger_{\alpha i}d^\dagger_{\alpha j}d_{j\beta}c_{i\beta}}{(k_3+k_4)^2}\delta(k_3+k_4-k_1-k_2) \\ &+ \frac{c^\dagger_{\alpha i}d^\dagger_{\alpha j}c^\dagger_{\beta j}d^\dagger_{\beta i}}{(k_3+k_4)^2}\delta(k_3+k_4+k_1+k_2) + \frac{c^\dagger_{\alpha i}d^\dagger_{\alpha j}c^\dagger_{\beta j}c_{i\beta}}{(k_3-k_4)^2}\delta(k_3-k_4+k_1+k_2) \\ &+ \frac{c^\dagger_{\alpha i}c_{j\alpha}d_{j\beta}d^\dagger_{\beta i}}{(k_3-k_4)^2}\delta(k_3-k_4-k_1+k_2) + \frac{c^\dagger_{\alpha i}c_{j\alpha}d_{j\beta}c_{i\beta}}{(k_3+k_4)^2}\delta(k_3+k_4-k_1+k_2) \\ &+ \frac{c^\dagger_{\alpha i}c_{j\alpha}c^\dagger_{\beta j}d^\dagger_{\beta i}}{(k_3+k_4)^2}\delta(k_3+k_4+k_1-k_2) + \frac{c^\dagger_{\alpha i}c_{j\alpha}c^\dagger_{\beta j}c_{i\beta}}{(k_3-k_4)^2}\delta(k_3-k_4+k_1-k_2)\Big) \end{aligned} \tag{2.32}$$

Ultimately, we are interested in the mass-squared operator $M^2 = 2P^+P^-$. In particular, we will eventually look for states that are eigenstates of M^2. This simply entails finding a simultaneous eigenbasis of P^+ and P^- since they commute.

2.5 Discretization

In the spirit of the first letter in DLCQ, we will work with a discretized version of this theory, in which the x^- direction is compactified to a circle of radius L. The fields $v, v^\dagger$ are antiperiodic on this circle. Thus in the Fourier expansions we must have $k = n/(2L)$ where n is a positive odd integer. Integrals over k are hence discretized into sums.

We define the dimensionless versions of the ladder operators by

$$D_{i\alpha} = \frac{1}{\sqrt{L}} d_{i\alpha}\big(\frac{n}{2L}\big),\ C_{i\alpha} = \frac{1}{\sqrt{L}} c_{i\alpha}\big(\frac{n}{2L}\big) \tag{2.33}$$

The Fourier expansions of the fields then become

$$v_{i\alpha} = \frac{1}{\sqrt{2\pi L}} \sum_{\mathbb{Z}_+^{\text{odd}}} \big(D_{i\alpha} e^{-inx^-/(2L)} + C^\dagger_{\alpha i} e^{inx^-/(2L)}\big) \tag{2.34}$$

$$v^\dagger_{\alpha i} = \frac{1}{\sqrt{2\pi L}} \sum_{\mathbb{Z}_+^{\text{odd}}} \big(D^\dagger_{\alpha i} e^{inx^-/(2L)} + C_{i\alpha} e^{-inx^-/(2L)}\big) \tag{2.35}$$

We impose the discretized anticommutation relations

$$\big\{D_{i\alpha}(n), D^\dagger_{\beta j}(m)\big\} = \big\{C_{i\alpha}(n), C^\dagger_{\beta j}(m)\big\} = \delta_{ij}\delta_{\alpha\beta}\delta_{nm} \tag{2.36}$$

with all other combinations being 0. The discretized momentum operators are

$$P^+ = \frac{1}{2L} \sum_{\mathbb{Z}_+^{\text{odd}}} n(D^\dagger_{\alpha i} D_{i\alpha} - C_{i\alpha} C^\dagger_{\alpha i}) = \frac{1}{2L} \sum_{\mathbb{Z}_+^{\text{odd}}} n(D^\dagger_{\alpha i} D_{i\alpha} + C^\dagger_{\alpha i} C_{i\alpha}) \tag{2.37}$$

$$P^-_{\text{mass}} = \frac{m^2 L}{2} \sum_{\mathbb{Z}_+^{\text{odd}}} \frac{1}{n}(D^\dagger_{\alpha i} D_{i\alpha} - C_{i\alpha} C^\dagger_{\alpha i}) = \frac{m^2 L}{2} \sum_{\mathbb{Z}_+^{\text{odd}}} \frac{1}{n}(D^\dagger_{\alpha i} D_{i\alpha} + C^\dagger_{\alpha i} C_{i\alpha}) \tag{2.38}$$

where in the 2nd steps we have gotten rid of the constant so that these operators give 0 on the vacuum state.

P^-_{int} eventually simplifies to

$$\begin{aligned} P^-_{\text{int}} = \frac{g^2 L}{\pi}\Big[& N \sum_{\mathbb{Z}_+^{\text{odd}}} \Big(\frac{1}{(n_1-n_2)^2} - \frac{1}{(n_1+n_2)^2}\Big)\Big(D^\dagger_{\alpha i}(n_1) D_{i\alpha}(n_1) + C^\dagger_{\alpha i}(n_1) C_{i\alpha}(n_1)\Big) \\ & + 2 \sum_{\mathbb{Z}_+^{\text{odd}}} \Big(\delta_{n_1+n_2, n_3+n_4} \big(\frac{C^\dagger_{\alpha i}(n_1) D^\dagger_{\alpha j}(n_2) D_{j\beta}(n_3) C_{i\beta}(n_4)}{(n_3+n_4)^2} - \frac{D^\dagger_{\alpha i}(n_1) C^\dagger_{\beta i}(n_2) C_{j\beta}(n_3) D_{j\alpha}(n_4)}{(n_2-n_3)^2} \big) \\ & + \frac{\delta_{n_1-n_2, n_3+n_4}}{(n_3+n_4)^2} \big(C^\dagger_{\alpha i}(n_1) C_{j\alpha}(n_2) D_{j\beta}(n_3) C_{i\beta}(n_4) + D^\dagger_{\alpha i}(n_1) D_{j\alpha}(n_2) C_{j\beta}(n_3) D_{i\beta}(n_4) \big) \\ & + \frac{\delta_{n_1+n_2, -n_3+n_4}}{(n_3-n_4)^2} \big(C^\dagger_{\alpha i}(n_1) D^\dagger_{\alpha j}(n_2) C^\dagger_{\beta j}(n_3) C_{i\beta}(n_4) + D^\dagger_{\alpha i}(n_1) C^\dagger_{\alpha j}(n_2) D^\dagger_{\beta j}(n_3) D_{i\beta}(n_4) \big) \Big) \Big] \end{aligned} \tag{2.39}$$

In the above, we have gotten rid of terms that are subleading in N or N_f, terms that must be 0 from momentum conservation, and normal ordering constants. We have also relabeled indices to combine terms.

2.6 Hilbert space

The next step is to construct the Hilbert space of this theory. We start with a normalized vacuum state $|0\rangle$ which is annihilated by all the $C_{i\alpha}, D_{j\beta}$ operators. We can get new "single-string" states of the following form:

$$|\psi\rangle \propto \prod_{j=1}^{p} C^\dagger_{\alpha_j i_j}(n_j) D^\dagger_{i_j \alpha_{j+1}}(m_j)|0\rangle \tag{2.40}$$

One can show that when $N \to \infty$ we have $\langle\psi|\psi\rangle = N^p N_f^{p-1}$, so the normalized single-string state is

$$|\psi\rangle = \frac{1}{N^{p/2} N_f^{(p-1)/2}} \prod_{j=1}^{p} C^\dagger_{\alpha_j i_j}(n_j) D^\dagger_{i_j \alpha_{j+1}}(m_j)|0\rangle \tag{2.41}$$

Such a state is manifestly SU(N)-invariant since it has no free gauge indices.

Let us consider the action of each term in P^+ on $|\psi\rangle$. The notation in the general case is a pain, so we explain the calculation in words. First consider just the $\sum_n n D^\dagger(n) D(n)$ term. Commuting the $D(n)$ all the way to the right of the string of creation operators splits $|\psi\rangle$ into a sum of states, since we get a new term every time D commutes past $D^\dagger$, and only one term in the sum over n contributes because of the Kronecker delta from the anticommutation relations. Now each string has two $C^\dagger$'s adjacent to each other; we commute the remaining $D^\dagger$ to land in between these, which results in all the negative signs flipping to positive. An example of such a calculation is worked out in Appendix B. In the end, we find that $|\psi\rangle$ is an eigenstate of P^+ with eigenvalue

$$\frac{K}{2L} = \frac{1}{2L} \sum_{j=1}^{p} \Big(\sum_{n_j \in \mathbb{Z}_+^{\text{odd}}} n_j + \sum_{m_j \in \mathbb{Z}_+^{\text{odd}}} m_j \Big) \tag{2.42}$$

There are also "multi-string" eigenstates which look like

$$|\chi\rangle \propto \prod_{k=1}^{r} \Big(\prod_{j_k=1}^{p_k} C^\dagger_{\alpha_{j_k} i_{j_k}}(n_{j_k}) D^\dagger_{i_{j_k} \alpha_{j_k+1}}(m_{j_k}) \Big)|0\rangle \tag{2.43}$$

where $r > 1$. We claim that, when properly normalized, $\langle\chi|P^-|\psi\rangle \to 0$ as $N \to \infty$, so we can diagonalize P^- with respect to only the single-string states. Moreover, $\langle\psi|\chi\rangle \to 0$ as well, so we can ignore the terms in P^-_{int} that generate multi-string states when acting on a single-string state. By induction on r we find that the normalized multi-string state is

$$|\chi\rangle = \prod_{k=1}^{r} \Big(\frac{1}{N^{p_k/2} N_f^{(p_k-1)/2}} \prod_{j_k=1}^{p_k} C^\dagger_{\alpha_{j_k} i_{j_k}}(n_{j_k}) D^\dagger_{i_{j_k} \alpha_{j_k+1}}(m_{j_k}) \Big)|0\rangle \tag{2.44}$$

and it has P^+-eigenvalue

$$\frac{K}{2L} = \frac{1}{2L}\sum_{k=1}^{r}\sum_{j_k=1}^{p_k}\Big(\sum_{n_{j_k}\in\mathbb{Z}_+^{\text{odd}}} n_{j_k} + \sum_{m_{j_k}\in\mathbb{Z}_+^{\text{odd}}} m_{j_k}\Big) \tag{2.45}$$

Note that in either case, K must be a positive even number, since the n_j's and m_j's come in pairs.

Let us consider a multi-string state $|\chi\rangle$ and a single-string state $|\psi\rangle$ which have the same eigenvalue $K/(2L)$ with respect to P^+. We now check that their inner product vanishes as $N\to\infty$ even though they live in the same degenerate eigenspace of P^+. First we note that if $p^\psi \neq \sum_k p_k^\chi$ then their inner product is exactly 0 because there would be a factor that looks like $\langle 1|0\rangle$. Thus we assume $p^\psi = \sum_k p_k^\chi = p$, and let r be the number of strings in $|\chi\rangle$. We get p pairs of Kronecker deltas with Latin indices summed over and $p-r$ pairs of Kronecker deltas with Greek indices summed over, since we lose a summed-over Greek index for each string. Without the normalization, the inner product gives $N^p N_f^{p-r} \approx N^{2p-r}$ factors of N. Including the normalization thus gives

$$\langle\psi|\chi\rangle \propto \frac{N^{2p-r}}{N^{2p}N^{-(r+1)/2}} = \frac{1}{N^{(r-1)/2}} \to 0 \tag{2.46}$$

Now consider the matrix element $\langle\psi|P^-|\chi\rangle$. Clearly the single- and multi-string states are also eigenstates of P^-_{mass}, so we immediately have $\langle\psi|P^-_{\text{mass}}|\chi\rangle\to 0$. What is left is to check that $\langle\psi|P^-_{\text{int}}|\chi\rangle\to 0$.

First let us assume $p^\psi = \sum_k p_k^\chi$. In this case only the terms in (2.39) with the same number of $D^\dagger$'s as D's and $C^\dagger$'s as C's will contribute. There are two types of terms of this form: (a) four operators and no factors of N, N_f; (b) two operators and one factor of N or N_f. Comparing to the inner product $\langle\psi|\chi\rangle$ computed earlier, we have one less factor of N or N_f for each pair of operators in the term, so in (a) we end up with 2 fewer factors of N and in (b) we end up with the same number of factors of N. Hence the overall product still goes to 0.

Alternatively, suppose $\sum_k p_k^\chi = p^\psi + 1$. Then the only contributing terms are $D^\dagger DDC$ and $C^\dagger CDC$. Neither of those terms have any factors of N or N_f, so the matrix element can have at most the same number of factors of N as $\langle\psi|\chi\rangle$, which implies it goes to 0 as desired.

A further restriction: we are interested in states that pair massless quarks with a massive antiquark or vice versa, i.e. exactly one of the free flavor indices at the ends of the string is equal to N_f. One can verify that these states form an invariant subspace of P^-. In particular, it suffices to look at states that end in a massive antiquark, since the matrix elements of these states with the states that start with a massive quark will be zero, and the two subspaces will have the exact same spectrum.

2.7 Correspondence with adjoint theory

We digress briefly to introduce a theory that is related to ours. Let us name our theory $\mathcal{T}'$ and the related theory $\mathcal{T}$, in accordance with the conventions of [7]. $\mathcal{T}$ is an SU(N) gauge theory whose matter content consists of a massive quark in the fundamental representation and a massless fermion in the adjoint representation of SU(N). In the $N\to\infty$

limit, pair creation is suppressed, so the fundamental quark is effectively non-dynamical and serves as a probe for the behavior of the pure adjoint theory.

We will not go too far into the details of this theory, but the important point is that $\mathcal{T}$ has operators $B^\dagger_{ij}$ that create adjoint quarks, as well as the $C^\dagger, D^\dagger$ operators that we have in $\mathcal{T}'$. This yields two types of gauge-invariant states: "meson" states which look like $C^\dagger_\alpha(B^\dagger \dots B^\dagger)D^\dagger_\beta|0\rangle$, and "gluinoball" states which look like $\mathrm{Tr}(B^\dagger \dots B^\dagger)|0\rangle$.

As mentioned in the Introduction, [7] conjectures that the spectrum of $\mathcal{T}$ is part of the spectrum of $\mathcal{T}'$, since the theories are thought to have primaries with the same KM level $k_{\mathrm{KM}} = N$ when $N_f = N + 1$. In particular, a single-string mesonic primary in the Kth subspace of $\mathcal{T}$ with massive quarks and one $B^\dagger(1)$ corresponds to a primary in $\mathcal{T}'$ obtained by replacing the $B^\dagger(1)$ with a $C^\dagger_{\alpha i}(1)D^\dagger_{\beta j}(1)$ pair, where α, β are massless flavors. This new primary is a two-string state in the $(K+1)$-th subspace of $\mathcal{T}'$ where the quarks on each end of the entire state are massive and all others are massless. The implication is that sums of two eigenvalues in the $(K+1)$-th subspace of the massive-massless sector of $\mathcal{T}'$ appear as eigenvalues in the Kth subspace of the massive sector in $\mathcal{T}$, although this does not recover all eigenvalues in $\mathcal{T}$. In the following section we will provide strong numerical evidence for this correspondence. A similar correspondence can be formulated for other sectors.

The lowest energy state in the massive-massless sector of $\mathcal{T}'$ corresponds to the start of the continuum in the massive sector of $\mathcal{T}$, which consist of mesons with two massive quarks. One can model such states via the interaction potential between the quarks as a function of their separation. As noted in the Introduction, qualitatively the potential looks like Figure 2. There exist bound states, and since the potential levels off at $r \to \infty$, there is also a continuum of states. Hence the lowest energy state in the continuum corresponds to the height of the interaction potential, which we will calculate numerically and extrapolate to $K \to \infty$ in Section 3.4. In other words, this is where the adjoint theory becomes screening, as opposed to confining.

3 Results

We diagonalize P^- in each subspace of K. The first step is to enumerate the states. We present a few low-dimensional examples. The only state with $K = 0$ is the vacuum state. There are $N_f - 1$ states with $K = 2$, given by

$$|\psi_{\gamma N_f}\rangle = N^{-1/2} C^\dagger_{\gamma k}(1) D^\dagger_{k N_f}(1)|0\rangle \tag{3.1}$$

where the Greek indices run from 1 to $N_f - 1$, i.e. the massless indices. For $K = 4$ we have

$$|\psi^1_{\gamma N_f}\rangle = N^{-1/2} C^\dagger_{\gamma k}(1) D^\dagger_{k N_f}(3)|0\rangle \tag{3.2}$$

$$|\psi^2_{\gamma N_f}\rangle = N^{-1/2} C^\dagger_{\gamma k}(3) D^\dagger_{k N_f}(1)|0\rangle \tag{3.3}$$

$$|\psi^3_{\gamma N_f}\rangle = N^{-1}(N_f - 1)^{-1/2} C^\dagger_{\gamma k}(1) D^\dagger_{k\zeta}(1) C^\dagger_{\zeta \ell}(1) D^\dagger_{\ell N_f}(1)|0\rangle \tag{3.4}$$

It turns out that, in all cases, P^- is diagonal in the flavor indices, so the only off-diagonal matrix elements we need to consider are the ones between states with different upper indices and the same flavor index. Thus for $K = 2$ we have a one-dimensional matrix, for $K = 4$ we have a three-dimensional matrix, etc.

K	# of states	K	# of states
0	1	16	2584
2	3	18	6765
4	8	20	17711
6	21	22	46368
8	55	24	121393
10	144	26	317811
12	377	28	832040
14	987	30	2178309

Table 1: Number of basis elements at each K, up to $K = 30$. These are alternating Fibonacci numbers.

In general, one finds that the dimension of these matrices is given by alternating Fibonacci numbers as shown in Table 1, which grow in size very quickly. We are able to obtain the full spectrum up to $K = 20$, and the lowest 100 eigenvalues for $20 < K \leq 30$, using Princeton's computing cluster and algorithms developed by [7, 12–15].

3.1 Spectrum and degeneracies with $N + 1$ flavors

For the purpose of comparing with $\mathcal{T}$, we set $N_f = N + 1$, and we diagonalize the dimensionless operator $\pi P^-/(g^2 LN)$ for convenience of notation and easier comparison with [7]. At $K = 2$ the matrix is one-dimensional, so the only eigenvalue is $y = \frac{m^2\pi}{g^2 N}$.

We are able to treat $K = 4$ analytically. The blocks of the matrix have the form

$$\frac{\pi}{g^2 LN} P^- = \begin{pmatrix} \frac{1}{2} + \frac{y}{3} & -1/2 & 1/2 \\ -1/2 & \frac{1}{2} + y & -1/2 \\ 1/2 & -1/2 & 1/2 + y \end{pmatrix} \tag{3.5}$$

We find the eigenvalues

$$\lambda_1 = y, \quad \lambda_{2\pm} = \frac{9 + 8y \pm \sqrt{81 + 24y + 16y^2}}{12} \tag{3.6}$$

which exactly match the prediction of $\mathcal{T}$ discussed in Section 2.7 and [7].

The $K = 6$ matrix is given by

$$\frac{\pi}{g^2 LN} P^- = \begin{pmatrix} 5/8 + y/5 & -1/8 & -1/2 & 1/2 & 1/8 & 1/8 & 0 & 0 \\ -1/8 & 5/8 + y & -1/2 & 0 & -1/8 & -1/8 & -1/2 & 0 \\ -1/2 & -1/2 & 1 + y/3 & -1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & -1/2 & 1 + y/3 & -1/2 & 0 & 0 & 1/2 \\ 1/8 & -1/8 & 0 & -1/2 & 5/8 + y & 1/8 & 0 & -1/2 \\ 1/8 & -1/8 & 0 & 0 & 1/8 & 5/8 + y & -1/2 & 1/2 \\ 0 & -1/2 & 1/2 & 0 & 0 & -1/2 & 1 + y & -1/2 \\ 0 & 0 & 0 & 1/2 & -1/2 & 1/2 & -1/2 & 1 + y \end{pmatrix} \tag{3.7}$$

When $y = 0$, the eigenvalues are

$$\{5/2, 3/2, 3/2, 1, 0, 0, 0, 0\} \tag{3.8}$$

When $y = 1$, the characterstic polynomial has a complicated cubic factor, so three of the roots cannot be written analytically in a compact form. The eigenvalues are

$$\{3.28639, 7/3, 2.15863, 1.75498, 1, 1, 1/2, 1/3\} \tag{3.9}$$

By setting $y = 1$ in (3.6) one can immediately observe that all the $K = 4$, $y = 1$ eigenvalues are repeated in at $K = 6$, $y = 1$. The eigenvalue 1 is doubly degenerate, which is expected because it also appears at $K = 2$, $y = 1$. Hence all eigenvalues from lower values of K are repeated with the same degeneracy as before.

We will not write down the higher-dimensional matrices, but let us list some $K = 8$, $y = 1$ eigenvalues so that we can exhibit the more subtle degeneracy patterns. There are 21 eigenvalues total, out of which the following 10 are rational:

$$\{5/2, 7/3, 7/3, 1, 1, 1, 1, 1/2, 1/2, 1/3\} \tag{3.10}$$

The eigenvalue $5/2$, which has not yet appeared at lower K, can be viewed as the sum of the "massive" eigenvalue 1 from $K = 2$, $y = 1$ and the "massless" eigenvalue $3/2$ from $K = 4$, $y = 0$.

The remaining 11 eigenvalues for $K = 8$, $y = 1$ are solutions to a cubic polynomial and a degree-eight polynomial, the latter of which cannot be solved by radicals. The numerical values are

$$\begin{aligned}\{&4.29735, 3.43134, 3.28639, 3.03279, 2.74248, 2.15863,\\ &1.87356, 1.79881, 1.75498, 1.36094, 0.250039\}\end{aligned} \tag{3.11}$$

We see that $\{2.15863, 1.75498, 3.28639\}$ all appear from $K = 6$, $y = 1$. The others are predicted by the correspondence with $\mathcal{T}$.

Plots of the spectra of P^- and M^2 with $N_f = N + 1$ and $y = 1$ are shown in Figures 3 and 4. We note the following degeneracy rules which have been alluded to previously:

1. At each K, the $y = 1$ eigenvalues from lower K are repeated, including degeneracy.
2. Sums of eigenvalues from lower K also appear at higher K, where one summand is with $y = 1$ and the others are with $y = 0$.

These patterns fit the predictions of the KM algebra construction. At each K, "new" eigenvalues not included in these patterns are predicted by the correspondence with $\mathcal{T}$.

3.2 Perturbation theory at small y

For $y \ll 1$, we can use perturbation theory to examine the y-dependence of the ground state at any finite K, i.e. treat P^-_{mass} as a small perturbation to P^-_{int}. Since the P^-_{int} eigenvalue 0 is highly degenerate, it will be useful to define some symmetry operators that commute with P^- in order to find the correct eigenstate. It turns out that there is a set of $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$ generators that satisfy this condition. These operators are defined in the adjoint theory $\mathcal{T}$, so we will not write them down explicitly. The one we

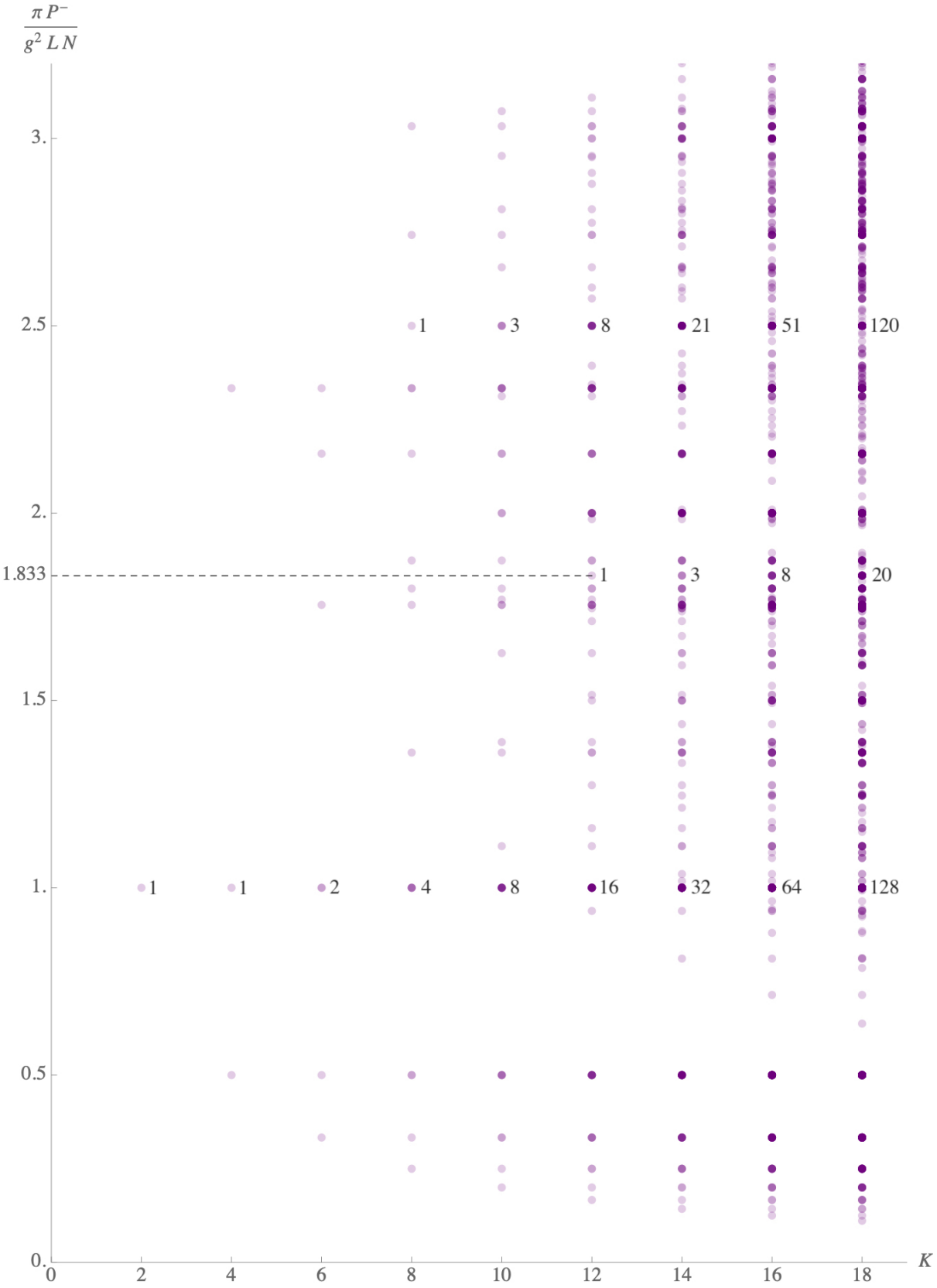


Figure 3: Spectrum of P^- with $N_f = N + 1$, $y = 1$ as discussed in Section 3.1. Opacity corresponds to the number of degeneracies. Some notable patterns are labeled.

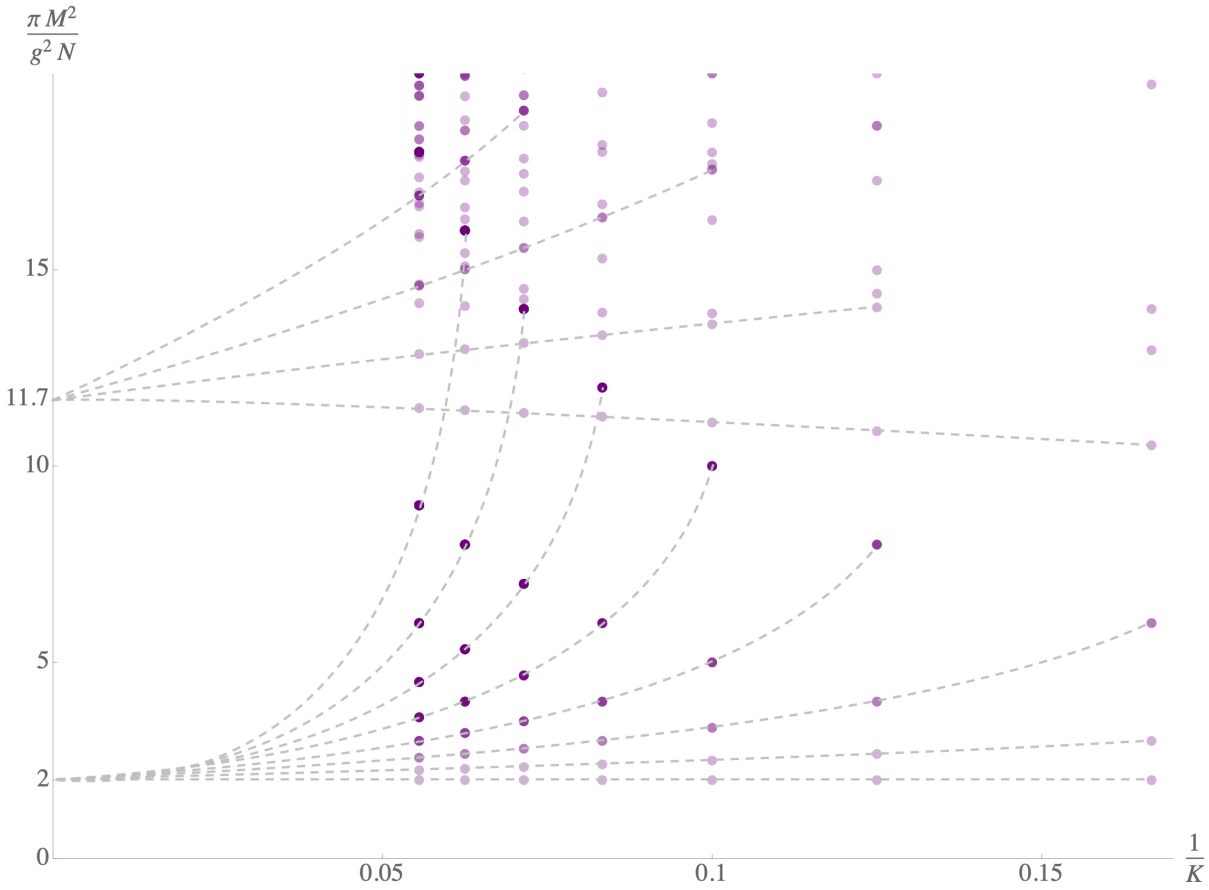


Figure 4: Spectrum of M^2 with $N_f = N + 1$, $y = 1$ as discussed in Section 3.1, again with opacity denoting degeneracies. As $K \to \infty$, the dotted trajectories converge towards the two lowest-mass states, $M^2 \approx 2.0\frac{\pi^2}{g^2 N}$ and $M^2 \approx 11.7\frac{\pi^2}{g^2 N}$ in agreement with the prediction of [7].

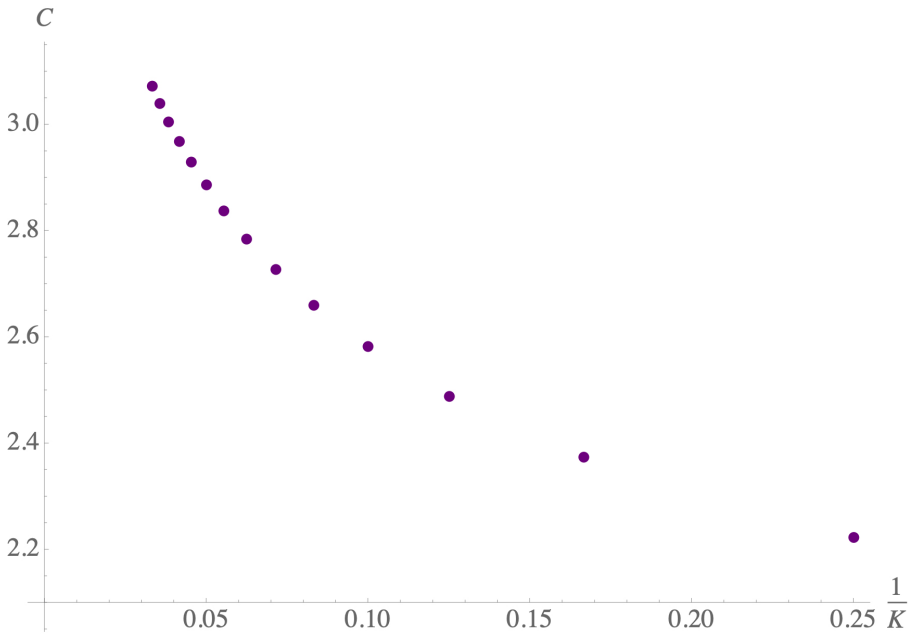
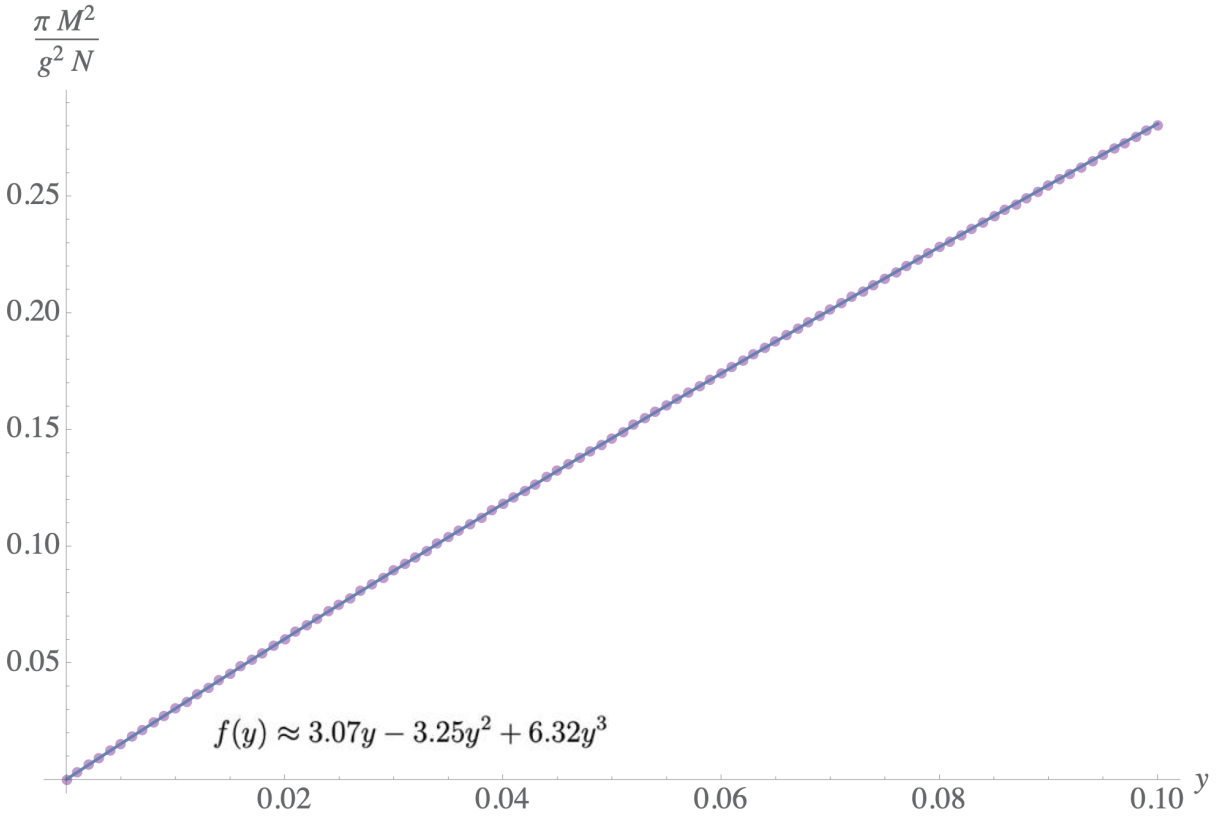
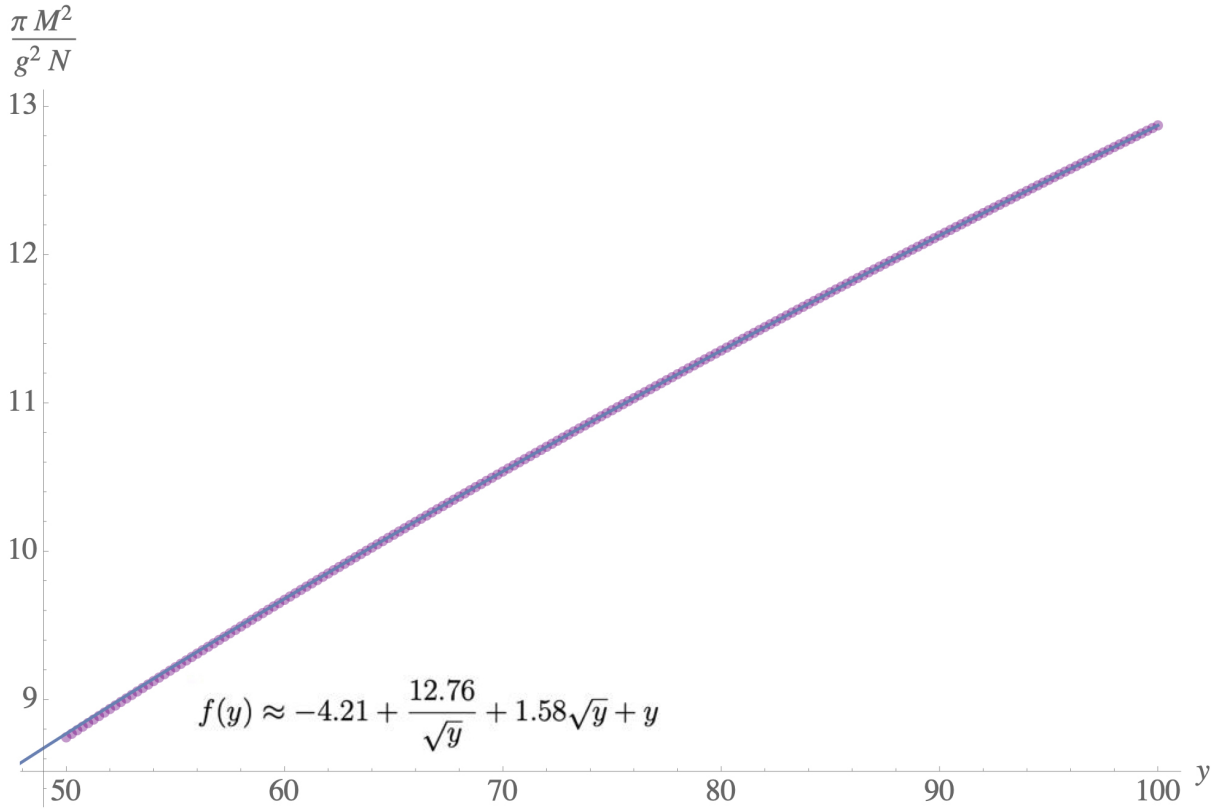


Figure 5: The first order perturbation $C(K) = \frac{\pi}{g^2 N}\langle\psi|M^2|\psi\rangle$ for $4 \leq K \leq 30$ and $N_f = N + 1$ as discussed in Section 3.2, plotted against $1/K$.



(a) Small y behavior as discussed in Section 3.2, fitted for $0 \leq y \leq 0.05$.



(b) Large y behavior as discussed in Section 3.4, fitted for $75 \leq y \leq 100$. The coefficient of y in the best fit is assumed to be 1.

Figure 6: Lowest eigenvalues of M^2 at $K = 30$ with $N_f = N + 1$.

are interested in acts on the left end of $\mathcal{T}'$ states, so let us call it J_L.

To be more concrete: at any fixed K, we can draw a correspondence between basis states in $\mathcal{T}'$ and mesonic basis states in $\mathcal{T}$ by taking a $\mathcal{T}'$ state and replacing all $C^\dagger$'s and $D^\dagger$'s with $B^\dagger$'s, except for the ones at each end, and contracting all the gauge indices. We can then compute the matrix elements of J_L in $\mathcal{T}$ with respect to this basis and check numerically that the resulting J_L matrix commutes with the P^- matrix in $\mathcal{T}'$ at each K. We do not yet know how to prove this fact, so it is an interesting question to study in the future.

The upshot of this discussion is that there exists a basis that simultaneously diagonalizes J_L and P^- at any finite K. The minimum possible eigenvalue of J_L is $1/4$, and in fact a simultaneous eigenstate of J_L and P^- has $J_L = 1/4$ if and only if its P^- eigenvalue does not appear in the spectrum at lower K; this is true regardless of the value of y in P^-_{mass}. Using this property, we can identify one unique state $|\psi\rangle$ with $J_L = 1/4, P^-_{\text{int}} = 0$, so it will not mix with other $P^-_{\text{int}} = 0$ states. This allows us to use nondegenerate perturbation theory. We used the computing cluster to find $|\psi\rangle$ and the first order perturbations $\langle\psi|P^-_{\text{mass}}|\psi\rangle$ up to $K = 30$. A plot of these values against $1/K$ is shown in Figure 5.

We can analytically derive the following formula for the perturbation at any K:

$$C(K) = \frac{\pi}{g^2 N}\langle\psi|M^2|\psi\rangle = K\frac{(K-2)!!}{(K-1)!!}\sum_{j=1}^{K/2}\frac{1}{K-2j+1}\frac{(2j-3)!!}{(2j-2)!!} \tag{3.12}$$

This agrees with the numerical results up to $K = 30$. Additionally, at $K = 30$, the best fit coefficient of y agrees with perturbation theory up to 0.014%. Figure 6a displays the numerically-determined best fit parameters for $y \ll 1$ at $K = 30$.

It turns out that $C(K)$ diverges logarithmically as $K \to \infty$, which is an indication that the leading order behavior at small y is not linear in the continuum case. An interesting question to study would be how to recover the exact y-dependence at small y from a perturbative series. Unfortunately, it is only possible to numerically compute higher order perturbation theory terms at relatively low K because one needs all the nonzero eigenvalues and corresponding eigenvectors of P^-_{int}. We are able to compute the second-order terms up to $K = 12$, but we do not have an analytic formula for general K.

3.3 Generalizing to more massless flavors

So far we have been setting $N_f = N + 1$. For spicier results we can generalize this to $N_f = aN + 1$ with $a \in \mathbb{Z}_+$. This only affects matrix elements of P^-_{int}. In particular, the $C^\dagger D^\dagger DC$ term in (2.39) will give a factor of a, and the terms with three creation operators or three annihilation operators will give a factor of $\sqrt{a}$.

As a result, the $K = 0, 2$ cases are unchanged. For $K = 4$ the matrix is given by

$$\frac{\pi}{g^2 LN}P^- = \begin{pmatrix} \frac{1}{2}+\frac{y}{3} & -1/2 & \sqrt{a}/2 \\ -1/2 & \frac{1}{2}+y & -\sqrt{a}/2 \\ \sqrt{a}/2 & -\sqrt{a}/2 & a/2+y \end{pmatrix} \tag{3.13}$$

and we find the eigenvalues

$$\lambda_1 = y, \quad \lambda_{2\pm} = \frac{6+3a+8y \pm \sqrt{9(2+a)^2+24ay+16y^2}}{12} \tag{3.14}$$

In general, the degeneracy patterns from Section 3.1 still hold. The higher a cases are useful for comparing to theories with more adjoint quarks. The perturbation theory calculations from Section 3.2 can be repeated as well.

3.4 Quark-antiquark potential in single- and multi-adjoint QCD

As noted several times, the upshot of studying the spectrum of $\mathcal{T}'$ is that we can compute the height of the interaction potential in $\mathcal{T}$. We do this by treating the lowest eigenvalue of M^2 as a function of y at each K and fitting a function

$$f(y) \approx y + b_1\sqrt{y} + b_2\sqrt{y}^{-1} + b_3 \tag{3.15}$$

for large y. The continuum in $\mathcal{T}$ begins at four times the lowest M^2 eigenvalue in $\mathcal{T}'$, so the coefficient b_1 in (3.15) corresponds to the height of the quark-antiquark potential $V_{\text{int}}(r)$ that describes the massive sector of $\mathcal{T}$, as explained in Section 2.7.

More explicitly, for a meson at the start of the continuum in $\mathcal{T}$ we have the following:

$$4M^2_{\mathcal{T}'} = M^2_{\mathcal{T}} = (2m_{\text{fund}} + V_\infty)^2 = \frac{4g^2N}{\pi}\left(y + \sqrt{\frac{\pi}{g^2N}}V_\infty\sqrt{y}\right) + V_\infty^2 \tag{3.16}$$

where $V_\infty = V_{\text{int}}(r \to \infty)$. This implies

$$V_\infty \approx \sqrt{\frac{g^2N}{\pi}}\, b_1 \tag{3.17}$$

at fixed K. To extrapolate to large K, we assume the function $b_1(K) \approx c_1 + c_2/K + c_3/K^2$. In the $K \to \infty$ limit, we find that b_1 approaches 1.3025, so V_{int} looks like Figure 7. Figure 6b displays best fit parameters at $K = 30$ for $y \gg 1$, and Figure 8a plots V_∞ in the best fits for $y \gg 1$ for each K.

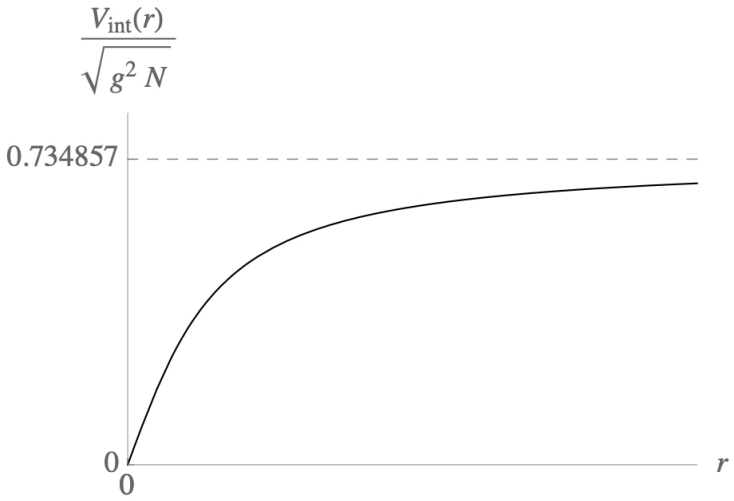
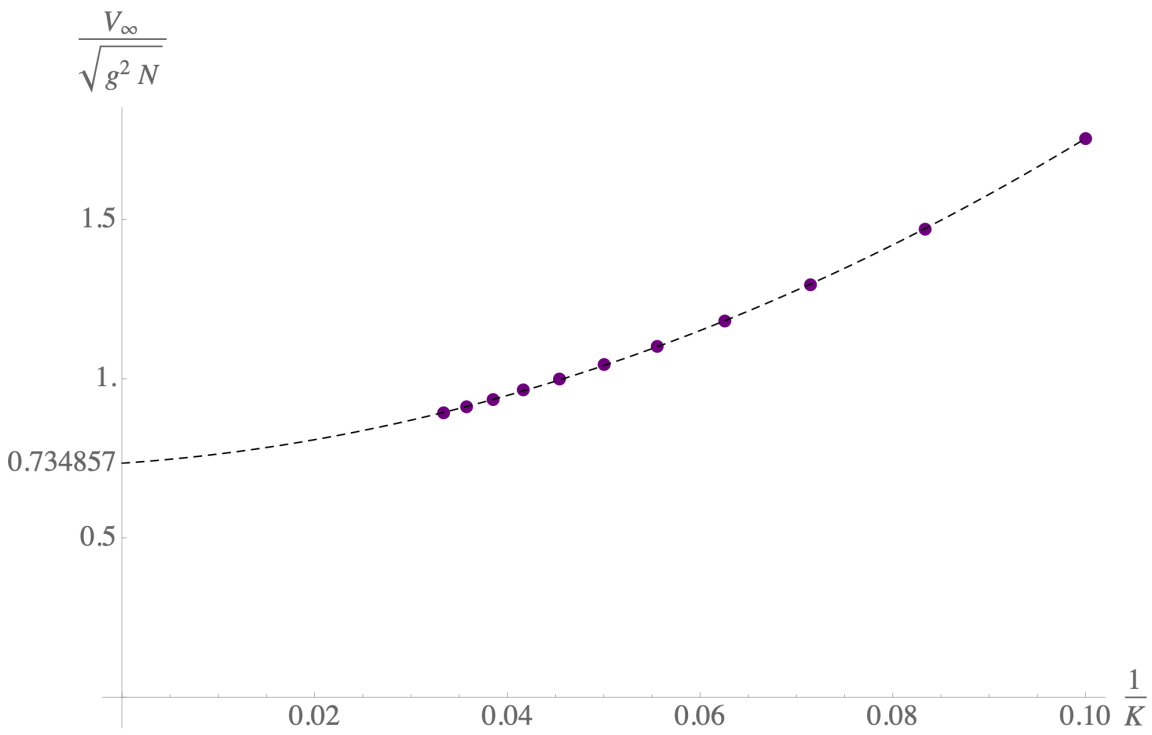
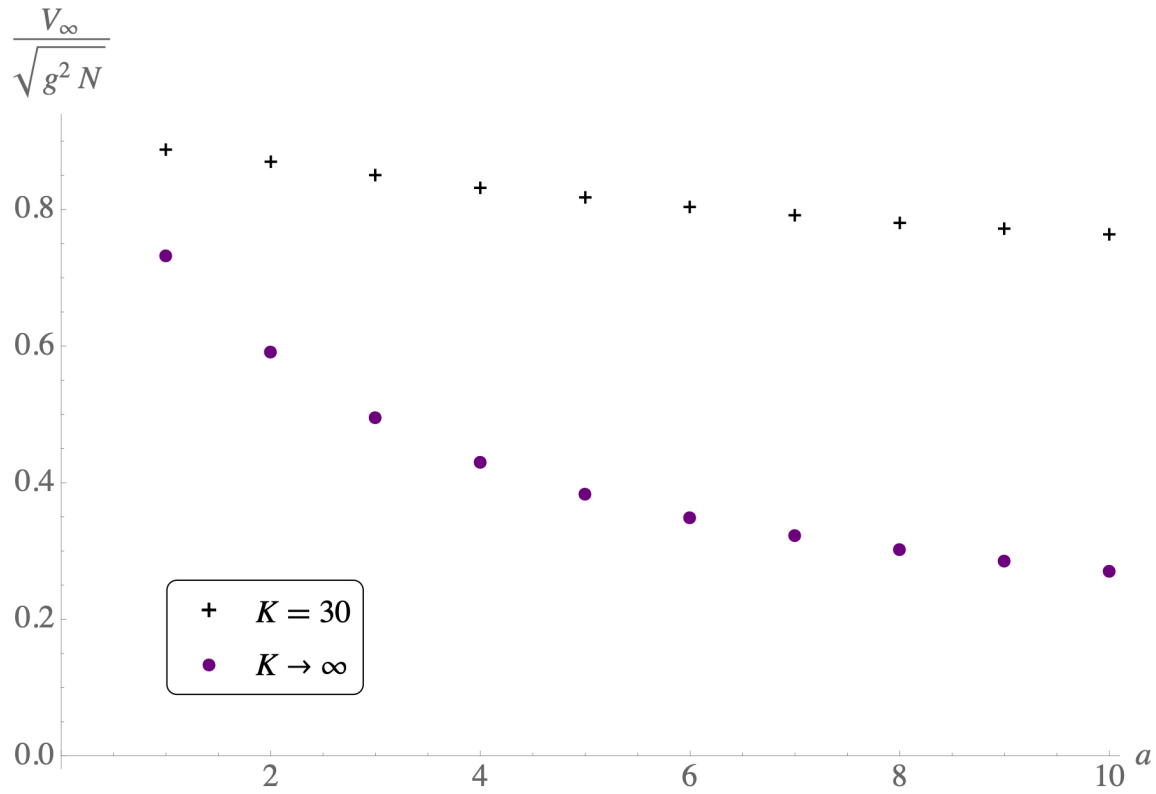


Figure 7: Quark-antiquark interaction potential V_{int} for mesons in $\mathcal{T}$ with extrapolated value $V_{\text{int}}(r \to \infty) \approx \sqrt{\frac{g^2N}{\pi}} \cdot 1.3025 \approx \sqrt{g^2N} \cdot 0.73486$.

The same procedure can be applied to the generalized theory with $N_f = aN + 1$, and it is interesting to see how $V_{\text{int}}(r \to \infty)$ varies with respect to a. We consider $a = 1, \ldots, 10$. Just as in the $a = 1$ case, these coefficients correspond to the height



(a) V_∞ plotted against $1/K$ with $a = 1$, yielding the extrapolated value $V_\infty(K \to \infty) \approx \sqrt{g^2 N} \cdot 0.73486$.



(b) V_∞ for $y \gg 1$ at $K = 30$ and $K \to \infty$, plotted for $1 \leq a \leq 10$.

Figure 8: V_∞ for $y \gg 1$, as discussed in Section 3.4. The best fits in the $K \to \infty$ extrapolations are assumed to be quadratic in $1/K$, i.e. $b_1(K) \approx c_1 + c_2/K + c_3/K^2$.

of the potential in the massive sector of the adjoint theory with a adjoint quarks. The extrapolated values for V_∞ as $K \to \infty$ are plotted with respect to a in Figure 8b; we have also included the values at $K = 30$ to show that the $K \to \infty$ limit gets further away from $K = 30$ as a increases, although the fit still works well.

4 Conclusion

In this paper, we numerically diagonalized one sector of $\mathcal{T}'$, an SU(N) gauge theory consisting of N_f flavors of fundamental quarks, using discretized light-cone quantization in the $N \to \infty$ limit. This gave rise to some interesting results. First, we showed that the spectrum of P^- in our theory with $N_f = N + 1$ corresponds to the spectrum of P^- in $\mathcal{T}$, another SU(N) gauge theory consisting of a massless adjoint and a massive fundamental quark, as predicted by the Kac-Moody construction. Using this relationship, we then used the large y behavior of the M^2 spectrum in $\mathcal{T}'$ to compute the height of the interaction potential at infinity for probe quarks in $\mathcal{T}$. We have also generalized our theory to $N_f = aN + 1$, which corresponds to adjoint theories with a massless adjoints.

Our perturbation theory calculation indicates that the $y \to 0$ and $K \to \infty$ limits do not commute, so finding the leading order y-dependence at small y in the continuum case will require extra care. It is likely that some combination of powers of y and $1/K$ needs to be kept finite as y and K are taken to infinity.

A possible direction for future research is to probe the interaction potential of $\mathcal{T}$ in regimes other than $r \to \infty$. One could also give the adjoint fermion a small mass and study the model perturbatively with respect to this. Such a theory would exhibit confining behavior and would not have any correspondence to $\mathcal{T}'$. In particular, the interaction potential in the massive adjoint theory would not approach a finite value at $r \to \infty$, so it would be interesting to extract the large r behavior.

Another possible direction is to study the same type of theory with a different structure group, such as SO(N) or Sp(N). One could also consider theories with different matter content, including fermions in representations of the structure group other than the fundamental and the adjoint. Such theories may exhibit interesting symmetries and degeneracies as well.

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Acknowledgements

I am beyond grateful to my advisor, Professor Silviu Pufu, for his generosity and patience while advising this paper. He taught me an immeasurable amount of physics over the course of the semester. I also thank graduate student Ross Dempsey for guidance on

many aspects of the project, from background topics to computer programs. I thank Professor Igor Klebanov for his insights as well, and for being the second reader on this paper.

I thank my fellow juniors Liza Rozenberg, Jesse Lin, Bryant Hall, and Cosmin Andrei for proofreading and feedback, as well as for moral support while I was chasing factors of two. I also acknowledge alumna Nicole Ozdowski for the LaTeX template.

Lastly, a special thanks goes to my roommate, Kaylin Xu, who is the sole reason I retained any semblance of sanity during this uniquely insane semester.

A Mathematical formalism

A.1 $\mathrm{SU}(N)$ trivia

$\mathrm{SU}(N)$ is defined as the group of $N \times N$ complex unitary matrices with determinant 1, where the group operation is simply matrix multiplication. For $N = 1$ it is the trivial group, and for $N > 1$ it is nonabelian.

One can show via the implicit function theorem that $\mathrm{SU}(N)$ is a (N^2-1)-dimensional real submanifold of $\mathrm{GL}(N, \mathbb{C})$, so it is a (N^2-1)-dimensional real Lie group with a manifold structure inherited from $\mathrm{GL}(N, \mathbb{C})$. A similar argument also shows that $\mathrm{SU}(N)$ is compact. Its Lie algebra $\mathfrak{su}(N)$ is generated by the $N \times N$ traceless Hermitian matrices (this differs from the mathematicians' convention by a factor of i). The Lie bracket is given by $-i$ times the commutator. If we denote the generators of $\mathfrak{su}(N)$ by $M_1, \ldots, M_{N^2-1}$, the commutation relations have the form $[M_a, M_b] = f_{abc}M_c$, and the structure constants f_{abc} are antisymmetric. These relations show that $\mathfrak{su}(N)$ is simple.

Since $\mathrm{SU}(N) \subset \mathrm{GL}(N, \mathbb{C})$, it acts naturally on the vector space $\mathbb{C}^N$ via left multiplication. Thus the inclusion map $\mathrm{SU}(N) \to \mathrm{GL}(N, \mathbb{C})$ gives an N-dimensional representation of $\mathrm{SU}(N)$. We call this the fundamental representation. It is an irreducible representation, or irrep for short, which means that its action on $\mathbb{C}^N$ has no proper nontrivial invariant subspaces. One can show this by considering the action of $\mathrm{SU}(N)$ on $S^{N-1} \subset \mathbb{C}^N$ and showing that it is transitive.

We can define another representation of $\mathrm{SU}(N)$ in the following way. We view $\mathfrak{su}(N)$ as a vector space and consider its automorphism group, i.e. linear endomorphisms of $\mathfrak{su}(N)$ which preserve the Lie bracket. Let $\rho : \mathrm{SU}(N) \to \mathrm{Aut}\big(\mathfrak{su}(N)\big)$ be the map that sends g to the $\mathfrak{su}(N)$ endomorphism $X \mapsto gXg^{-1}$. This is an (N^2-1)-dimensional representation known as the adjoint representation, and it is irreducible since $\mathfrak{su}(N)$ has no nontrivial proper ideals.

A.2 Basics of gauge theory

The basic idea of a gauge theory is the following: we start with a principal bundle $\pi : P \to M$ with structure group G and an affine connection A on P. Next, we take some representation $\rho : G \times V \to V$ of G and form the associated vector bundle $E = (P \times V)/\sim$, where $\sim$ is the equivalence relation $(p, v) \sim (gp, g^{-1}v)$ and the projection is $\tilde{\pi} : \overline{(p, v)} \mapsto \pi(p)$.

This construction naturally describes field theories that have certain local symmetries called gauge symmetries. Namely, we fix a principal bundle P over M with structure

group G and associated vector bundle E, and define a field q as a section of the fibers of the vector bundle, i.e. $q : M \to E$ such that $\tilde{\pi} \circ q = \mathrm{id}_M$. In physics, M is spacetime; we can think of E as the "state space" of each point in spacetime. Now if we introduce a gauge field A_μ, which plays the role of the connection on the bundle, we have a way of identifying the state spaces of different points in M by considering parallel transport along some path in M. In general, parallel transport along two different paths is only unique up to action of G on E; this is described by the curvature $F(v, w) = \nabla_v \nabla_w - \nabla_w \nabla_v - \nabla_{[v,w]}$ where ∇ is the covariant derivative induced by A. The curvature two-form $F_{\mu\nu}$ is the field strength tensor from physics. We can think of it as measuring the "distortion" of the total state space induced by the gauge field.

The all-important notion in gauge theory is the idea of gauge transformations, which are simply bundle automorphisms $\varphi : E \to E$ such that $\tilde{\pi} \circ \varphi = \varphi \circ \tilde{\pi}$. The set of all gauge transformations $\mathcal{G}$ of a bundle forms a group where the group operation is pointwise multiplication. Physicists usually call $\mathcal{G}$ the gauge group (not to be confused with the structure group G, which mathematicians call the gauge group). A gauge transformation of E induces a transformation on the connection; we say that two connections A, A' are gauge equivalent if their pullbacks on M are related by $A'_\mu = gA_\mu g^{-1} + g\partial_\mu g^{-1}$ for some $g \in \mathcal{G}$. In physics, we interpret gauge transformations as leaving the physical system unchanged. In other words, we postulate that all physical quantities must be invariant under gauge transformations, i.e. gauge invariant.

B An example calculation

As a simple example, we compute $P^+|\psi\rangle$ where $|\psi\rangle = C^\dagger_{\gamma j}(k_1) D^\dagger_{j\zeta}(k_2) C^\dagger_{\zeta\ell}(k_3) D^\dagger_{\ell\rho}(k_4)|0\rangle$. We first compute the $D^\dagger D$ term in (2.30):

$$\sum_n n D^\dagger_{\alpha i}(n) D_{i\alpha}(n)|\psi\rangle = \big(- k_2 D^\dagger_{\alpha i}(k_2) C^\dagger_{\gamma j} \delta_{ij} \delta_{\alpha\zeta} C^\dagger_{\zeta\ell} D^\dagger_{\ell\rho} + k_4 D^\dagger_{\alpha i}(k_4) C^\dagger_{\gamma j} D^\dagger_{j\zeta} C^\dagger_{\zeta\ell} \delta_{i\ell} \delta_{\alpha\rho} \big)|0\rangle \tag{B.1}$$

$$= \big(k_2 C^\dagger_{\gamma j} D^\dagger_{j\zeta}(k_2) C^\dagger_{\zeta\ell} D^\dagger_{\ell\rho} + k_4 C^\dagger_{\gamma j} D^\dagger_{j\zeta} C^\dagger_{\zeta\ell} D^\dagger_{\ell\rho}(k_4) \big)|0\rangle = (k_2 + k_4)|\psi\rangle \tag{B.2}$$

where in intermediate steps we have suppressed the arguments of the operators in $|\psi\rangle$ for easier notation. Schematically, one can think of the action of $D^\dagger(n)D(n)$ as replacing each $D^\dagger$ in $|\psi\rangle$ with the $D^\dagger$ in P^+ and setting all the indices equal to those of the operator that was replaced.

A similar calculation shows

$$\sum_n n C^\dagger_{\alpha i}(n) C_{i\alpha}(n)|\psi\rangle = (k_1 + k_3)|\psi\rangle \tag{B.3}$$

so we finally have

$$P^+|\psi\rangle = \frac{1}{2L}(k_1 + k_2 + k_3 + k_4)|\psi\rangle \tag{B.4}$$

as desired.

The procedure for calculating $P^-|\psi\rangle$ is very similar. The main difference is that P^-_{int} has four-operator terms that all have different arguments, so one ends up with a sum over n in those cases. Taking the inner product with another state then picks out one term in this sum.